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BOOLEAN ALGEBRA

1. DEFINITION

A Boolean algebra is an algebraic structure which consists of a non empty set B, equipped with two binary operations (denoted by $\vee$ and $\wedge$ or + and $\bullet$ or * and o), one unary operation (denoted by /) and two specially defined elements 0 and I (in B) and which satisfy the following five laws for all values of a, b, c, $\in$ B.

1. Closure: The set B is closed with respect to each of the two operations i.e.

$$a \vee b \text{ or } a + b \in B \;\forall\; a, b \in B$$

and

$$a \wedge b \text{ or } a \bullet b \in B \;\forall\; a, b \in B$$

This property is satisfied by virtue of the two operations being binary operations. Also $a \in B \Rightarrow a' \in B$.

2. Commutative laws:

(i) $a \vee b = b \vee a$ or $a + b = b + a$

(ii) $a \wedge b = b \wedge a$ or $a \bullet b = b \bullet a$

3. Associative laws:

(i) $a \vee (b \vee c) = (a \vee b) \vee c$ or $a + (b + c) = (a + b) + c$

(ii) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$ or $a \bullet (b \bullet c) = (a \bullet b) \bullet c$

4. Distributive laws:

(i) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ or $a + (b \bullet c) = (a + b) \bullet (a + c)$

(ii) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ or $a \bullet (b + c) = a \bullet b + a \bullet c$

5. Identity laws:

(i) $a \vee 0 = a = 0 \vee a$ or $a + 0 = a = 0 + a$

$a \wedge I = a = I \wedge a$ or $a \bullet I = a = I \bullet a$

Here 0 is the identity for the operation $\vee$ or + (called **join** or sum) and I is the identity element for the operation $\wedge$ or $\bullet$ (called **meet** or product) 0 is called the **zero element** and I is called the **unit element** of the Boolean algebra.

5. Complementation or Domination laws:

(i) $a \vee a' = I$ or $a + a' = I$ (the identity for $\wedge$ or $\bullet$ operation)

(ii) $a \wedge a' = 0$ or $a \bullet a' = 0$ (the identity for $\vee$ or + operation)

It should be noted here that a′ is the complement of a for both the operation (i.e. $\vee$ and $\wedge$ or + and $\bullet$). It means that the complement of an element $a \in B$ is the same for both the binary operations.

We write this algebraic structure as (B, $\vee$, $\wedge$, /, 0, I) or (B, +, $\bullet$, /, 0, I) or simply B.

The binary operation $\vee$ or + is called **join** or sum operation while the binary operation $\wedge$ or $\bullet$ is called **meet** operation. The unary operation / is called **complementation** operation.

0 and I are the **identities** for **join** and **meet** and, **product** operations respectively.

Note: Sometimes the dot operation (•) is also omitted and we write a b to denote a • b or a $\wedge$ b where no confusion arises.

Unless guided by the presence of parentheses, the opeation ‘(complementations) has precedance over the operation • (or $\wedge$ or o) called product or meet and the operation • (or $\wedge$ or o) has precedence over the operation + (* or $\vee$) called sum or join during simplification.

Illustration: 1 $x + y \cdot z = x + (y \cdot z)$

and, $x + y \cdot z \neq (x + y) \cdot z$

2. $x \cdot y' = x \cdot (y')$

and, $x \cdot y' \neq (x \cdot y)'$

Example 1: Let (B, +, •) be an algebraic structure. + and • are two operations for the set B = {0, 1} defined as follows

+	0	1
0	0	1
1	1	1

•	0	1
0	0	0
1	0	1

Prove that the given set B with defined binary operations is a Boolean algebra.

Solution: **1. Closure property** is satisfied as all the elements in the two tables belong to B, i.e. The set B is closed with respect to the two binary operations.

2. Commutativity property is also satisfied because of the symmetry in both the tables about the leading diagonals.

3. Associativity property is also satisfied as we can see from the table that (1 + 0) + 1 = 1 + 1 = 1 and 1 + (0 + 1) = 1 + 1 = 1.

Consequently 1 + (0 + 1) = (1 + 0) + 1

Thus associativity is satisfied for + operation

Also (1 • 0) • 1 = 0 • 1 = 0 and 1• (0 • 1) = 1 • 0 = 0

Consequently (1 • 0) • 1 = 1 • (0 • 1)

Thus associativity is satisfied for • operation.

4. Distributive property is also satisfied as we can see from the table that

$$\left.\begin{array}{l} 1 + (0 \cdot 1) = 1 + 0 = 1 \\ \text{and} \quad (1 + 0) \cdot (1 + 1) = 1 + 1 = 1 \end{array}\right\} \quad \text{giving } 1 + (0 \cdot 1) = (1 + 0) \cdot (1 + 1)$$

i.e. + distributes over •

$$\left.\begin{array}{l} \text{Again} \quad 1 \cdot (0 + 1) = 1 \cdot 1 = 1 \\ \text{and} \quad (1 \cdot 0) + (1 \cdot 1) = 0 + 1 = 1 \end{array}\right\} \quad \text{giving } 1 \cdot (0 + 1) = (1 \cdot 0) + (1 \cdot 1)$$

i.e. • distributes over +

5. Identity laws: As 0 + 0 = 0 and 1 + 0 = 1 = 0 + 1. Therefore 0 is the identity for + operation. Again 0•1 = 0 = 1•0 and 1•1 = 1 = 1•1. Therefore 1 is the identity for • operation, Unique identities exist for both the operations.

6. Complementation laws: 0 + 1 = 1 = 1 + 0

0•1 = 0 = 1•0

Therefore 1 is the complement of 0 for both operations.

Also 1 + 0 = 1 = 0 + 1

and 1•0 = 0 = 0•1

Therefore 0 is the complement of 1 for both operations. Hence complements exist for each element. Therefore (B, +, •) is Boolean algebra.

Other Illustrations of Boolean Algebra

1. If S is a non empty set, then the power set P(S) of S along with the two operations of **union** and **intersection** i.e. (P(S), $\cup$, $\cap$) is a Boolean algebra. We know that the two operation $\cup$ and $\cap$ satisfy the closure, associativity, commutativity and distributivity properties $\forall$, A, B, C $\in$ P(S). Also the null set $\varnothing$ is the identity for the operation of union and the universal set U is the identity for the operation of intersection. Also there exists a complement $A' = U - A$ for each element $A \in P(S)$ for both the operations.

2. The set S of propositions together with binary operations **conjunction** ($\wedge$) and **disjunction** ($\vee$) and unary operation of negation is a Boolean algebra. F, the set of contradiction is identity for the operation of disjunction ($p \vee F = p$) and T, the set of tautologies is the identity for the operation of conjunction ($p \wedge T = p$). Also complement of p is p′ for both operation as

$p \wedge P' = F$ (identity for $\vee$ operation)

$p \vee p' = T$ (identity for $\wedge$ operation)

3. Let $D_{24} = \{1, 2, 3, 4, 6, 8, 12, 24\}$, the divisors of 24.

If we define the operations +, • and/on D_{24} as:

$$a + b = \text{l.c.m.}(a, b), \quad a \bullet b = \text{g.c.d}(a, b)$$

$$a' = \frac{24}{a} \text{ for any } a, b \in D_{24}, \text{ then}$$

D_{24} is a Boolean algebra with 1 as the zero element and 24 as the unit element.

2. RELATIONSHIP BETWEEN BOOLEAN ALGEBRA AND LATTICE (Boolean algebra as lattice)

A lattice L is a partially ordered set in which every pair of elements $x, y \in L$ has a least upper bound denoted by l u b (x, y) and a greatest lower bound denoted by g l b (x, y).

The two operations of meet and join denoted by $\wedge$ and $\vee$ respectively defined for any pair of elements $x, y \in L$ as

$$x \vee y = \text{l u b}(x, y) \text{ and } x \wedge y = \text{g l b}(x, y)$$

A lattice L with two operations of meet and join shall be a Boolean algbera if L is

1. Complemented: i.e. (i) if must have a least element 0 and a greatest element 1.

and (ii) For every element $x \in L$, these must exist an element $x' \in L$ such that

$$x \vee x' = 1 \text{ and } x \wedge x' = 0$$

2. Distributed: i.e. $\forall\, x, y, z \in L$

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

and, $$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

A Boolean algebra is a complemented, distributive lattice. It is generally denoted by (B, +, •, ′, 0, 1). Here (B, +, •) is a lattice with two binary operations + and • called the join and meet respectively. The corresponding poset is represented by (B, $\leq$) whose least and the greatest elements are denoted by 0 and 1 that are also the lower and upper bounds of the lattice. (B, +•) being a complemented, distributive lattice, each element of B has a unique complement. Complement of a is denoted by a′.

Theorem 1: The following are equivalent expressions in Boolean algebra:

(i) $a + b = b$ (ii) $a \bullet b = a$ (iii) $a' + b = 1$ (iv) $a + b' = 0$

Whenever any one of the above four condition is true we can say that $a \leq b$ (a precedes b).

We shall illustrate this theorem with the help of following two illustrations:

Illustration 1: If $\{P(S), \cup, \cap, \sim, \phi, U\}$ is a Boolean algebra of sets, then a set $A \in P(S)$ preceds another set $B \in P(S)$ if $A \subseteq B$. Therefore, according to this theorem, if $A \subseteq B$

(i) $A \cup B = B$ (ii) $A \cap B = A$ (iii) $A' \cup B = U$ (iv) $A \cap B' = \phi$

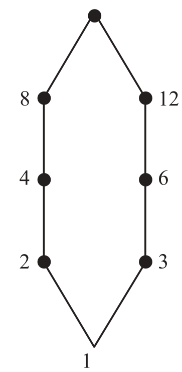


Illustration 2: Let D_{24} = {1, 2, 3, 4, 6, 8, 12, 24} be the set of divisors of 24. We say that $a \leq b$ (a precedes b) if a divides b.

Then l.c.m. (a, b) = b and g.c.d. (a, b) = a as shown below:

l.c.m. (2, 6) = 6 and g.c.d (2, 6) = 2

l.c.m (2′, 6) = l.c.m (12, 6) = 12 = l.u.b.

g.c.d (2, 6′) = g.c.d (2, 4) = 2 = g.l.b.

Theorem 2: If an integer n can be written as product of r distinct primes i.e.

$$n = p_1 \cdot p_2 \ldots p_r$$

where p_i s are distinct primes, then the lattice D_n, the divisors of n under the operations of meet and join $(\wedge$ and $\vee)$ is a Boolean algebra.

Illustration: D_{33} is a Boolean algebra because 33 = 3.11

D_{105} is a Boolean algebra because 105 = 3.5.11

D_{70} is a Boolean algebra because 70 = 2.5.7

D_{40} is not a Boolean algebra because 40 = 2.2.2.5 as three 2s are not distinct prime

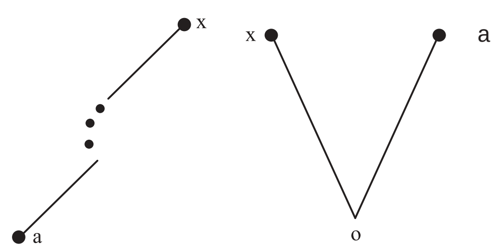
D_{75} is not a Boolean algebra because 75 = 3.5.5 as two 5s are not distinct primes.

3. ATOM

A non zero element a in a Boolean algebra (B, +, •, ′, 0, 1) is called an atom if $\forall x \in B$,

(i) $x \wedge a = a$ (i.e. x is a successor of a) as shown in fig. (i)

or (ii) $x \wedge a = 0$ (i.e. x and a are not related) as shown in fig. (ii).



(i) (ii)

Hence, we can say that if (B, +, •, ′, 0, 1) is a Boolean algebra, then an element $a \in B$ is an atom if a immediately succeeds 0 ie. 0 << a.

Illustration 1: Let $(B, \wedge, \vee, ', 0, 1)$ be a Boolean algebra where B = {1, 2, 3, 5, 6, 10, 15, 30} = D_{30} operations $\wedge$ and $\vee$ denotes g.c.d and l.c.m. respectively. The relation $\leq$ is 'divides'. Zero element is 1.

Then the set of atoms of the Boolean algebra i.e. set of elements that are immediate successors of the zero element 1 is {2, 3, 5}.

Illustration 2: Let P(A) be the power set of a set A. Then $(P(A), \cup, \cap, ')$ is the Boolean algebra. If the relation $\leq$ is set inclusion $(\subseteq)$, then the singleton sets are the atoms and every element in P(A) can be represented completely and uniquely as the union of singleton sets.

Example 2: Find atoms of the Boolean algebras

(i) B^2 (ii) B^4 (iii) $B^n, n \geq 1$

where B = {0, 1} and set of binary digits with the set of binary operations + and •, and unary operation ′ are given as follows:

+	0	1
0	0	1
1	1	1

•	0	1
0	0	0
1	0	1

′	1
1	0
0	1

Solution: (i) $B^2 = B \times B = \{(0,0), (0,1), (1,0), (1,1)\}$

The operation +, and ′ can be defined for B^2 as follows:

$$(0,1) + (1,1) = (0 + 1, 1 + 1) = (1,1)$$

$$(0,1) \cdot (1,1) = (0 \cdot 1, 1 \cdot 1) = (0,1)$$

$$(0,1)' = (0', 1') = (1,0)$$

Therefore, B^2 is a Boolean algebra of order 4.

The atoms in B^2 are (0, 1) and (1, 0).

(ii) The atoms are (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0) and (0, 0, 0, 1).

(iii) The atoms are n-tuples with exactly one 1.

Representation Theorem 3: Let A be the set of atoms of B and let P(A) be the Boolean algebra of all subsets of the set A then each $x \neq 0$ in B can be expressed uniquely (except for order) as the sum (or join) of atoms (= elements of A) like

$$x = a_1 + a_2 + \ldots + a_n$$

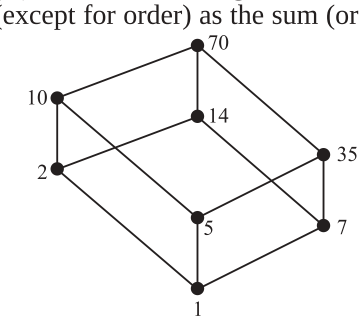
Then the unique mapping $f : B \to P(A)$ defined as

$$f(x) = \{a_1, a_2, \ldots, a_n\} \text{ is an isomorphism.}$$

Illustration: Let us consider the Boolean algebra

$$D_{70} = \{1, 2, 5, 7, 10, 14, 3, 5, 70\}$$

$$A = \{2, 5, 7\} \text{ is the set of atoms of } D_{70}.$$



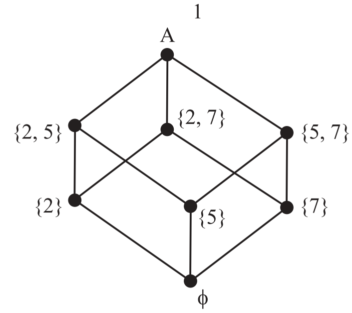
We can represent each of the non atom by atoms as shown below:

$10 = 2 + 5$ or $2 \vee 5$

$14 = 2 + 7$ or $2 \vee 7$

$35 = 5 + 7$ or $5 \vee 7$

$70 = 2 + 5 + 7$ or $2 \vee 5 \vee 7$



4. SUB-BOOLEAN ALGEBRA

Let B be a non empty set and (B, +, •, ′) a Boolean algebra with 0 and I as identity elements for the binary operations + and • respectively. Let B′ be a non empty subset of B.

If B′ contains the elements 0 and 1 and is closed under the operations +, • and ′, then (B′, +, •, ′) or (B′, +, •, ′, 0, 1) is called a sub-Boolean algebra or sub-algebra. It is evident that B′ itself is a Boolean algebra with respect to the operations of B. If we want to check whether B′ is closed under the three operations +, • and ′, and also to check whether 0 and 1 are in B′, then it is sufficient for these purposes that we check whether B′ is closed either with respect to the set of operations {+, ′} or {•, ′} only. It means that if B′ is closed under the operation + and ′ or under the operations • and ′ then B′ is a sub-Boolean algebra. This is possible because these sets of operations are functionally complet due to the following properties:

$\forall\, a, b \in B$

$a + b = (a' \bullet b')'$ …(1)

which means that if B is closed under • and ′ then for

$a, b \in B \Rightarrow a', b' \in B$ and $a' \bullet b' \in B$ and $(a' \bullet b')' \in B$

and therefore, a + b which is equal to $(a' \bullet b')'$ also belongs to B i.e. B is closed under + also.

Similarly we can show that if B is closed under + and ′, then B is closed under $\bullet$ also.

Again $1 = (a \bullet a')'$ and $0 = a \bullet a'$ …(2)

which means that if $a \in B$ and B is closed under the set of operations {$\bullet$, ′} then

$(a \bullet a') \in B \Rightarrow 0 \in B$

and $(a \bullet a')' \in B \Rightarrow 1 \in B$

Note: 1. The subset {0, 1} and the set B are both sub-Boolen algebras.

2. The set {a, a′, 0, 1} where $a \neq 0$ and $a \neq 1$, is a sub-Boolean algebra of the Boolean algebra (B′, +, $\bullet$, ′, 0, 1).

3. Any subset of B generates a sub-Boolean algebra.

Exercise (based on above decision): Define sub-algebra. Prove that a non-empty subset S of a Boolean algebra B is a sub algebra of B, iff S is closed with respect to operations + and ′ (addition and complementation). **[C.C.S.U., M.Sc. (Maths) 2004]**

Theorem 4: If S_1 and S_2 are two sub-algebras of a Boolean algebra B, then prove that $S_1 \cap S_2$ is also a sub-algebra of B.

Proof: Here we have to prove that $S_1 \cap S_2$ is closed with respect to the set of operations {+, ′}.

Let $a, b \in S_1 \cap S_2$.

It implies that:

(i) $a, b \in S_1 \Rightarrow (a + b) \in S_1$ (as S_1 is sub algebra) and, …(1)

(ii) $a, b \in S_2 \Rightarrow (a + b) \in S_2$ (as S_2 is sub algebra) …(2)

(1) and (2) imply that

$a, b \in S_1 \cap S_2 \Rightarrow (a + b) \in S_1 \cap S_2$

Therefore $S_1 \cap S_2$ is closed with respect to the operation +.

Again let $a \in S_1 \cap S_2$

$\Rightarrow$ $a \in S_1 \Rightarrow a' \in S_1$ (as S_1 is sub algebra) …(1)

Also,

$\Rightarrow$ $a \in S_2 \Rightarrow a' \in S_2$ (as S_2 is sub algebra) …(2)

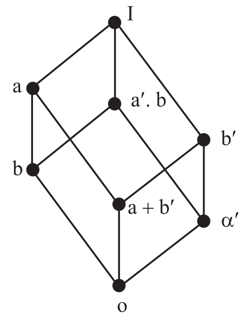
(1) and (2) imply that $a' \in S_1 \cap S_2$

It means that $a \in S_1 \cap S_2, a' \in S_1 \cap S_2$

or $S_1 \cap S_2$ is closed with respect to the operation ′ (Complementation)

Thus $S_1 \cap S_2$ is closed with respect to the set of operation {+, ′} and hence is a sub algebra

Illustration: Let the Boolean algebra be expressed by the following figure.



The following two subsets are sub-Boolean algebras

$B_1 = \{a, a', 0, 1\}$

$B_2 = \{a', \bullet b,\ a + b', 0, 1\}$

The following two subsets are Boolean algebras but not sub Boolean algebras

$B_3 = \{a + b',\ b', a, 1\}$

$B_4 = \{b',\ a + b', a', 0\}$

The subset $B_5 = \{a, b', 0, 1\}$ is not a Boolean algebra and hence is not a sub-Boolean algebra.

Illustration: Let B = D (30) = {1, 2, 3, 5, 6, 15, 30} be a Boolean algebra [as 30 = 2.3.5. is a product of distinct primes].

The subsets B_1 = {1, 2, 15, 30}, B_2 = {1, 3, 10, 30}, B_3 = {1, 5, 6, 30} are Boolean subalgeba of B.

The following subsets of B are Boolean algebras but not Boolean subalgebras of B. These sets are sub-lattices of D(30).

S_1 = {1, 2, 3, 6} S_4 = {2, 6, 10, 30}

S_2 = {1, 3, 5, 15} S_5 = {5, 10, 15, 30}

S_3 = {1, 2, 5, 10} S_6 = {3, 6, 15, 30}

The following subsets are not Boolean algebras but these are sub lattices

S_1 = {1, 3, 6, 15, 30}, S_2 = {1, 2, 3, 6, 15, 30}

5. IDEALS OF BOOLEAN ALGEBRA

Let (B′, +, •, ′) be a Boolean algebra with 0 and I as identity elements for the operations + and • respectively and B′ be a non-empty subset of B. Then, B′ is called Ideal of B if

(i) $a, b \in B' \Rightarrow a + b \in B', \forall\, a, b$

(ii) $a \in B', s \in B \Rightarrow a \bullet s \in B'$

Theorem 5: Intersection of two ideals of a Boolean Algebra B is also an ideal.

Proof: Let B_1 and B_2 be two ideals of a Boolean algebra B. It is required to prove that $B_1 \cap B_2$ is an ideal of B.

As B_1 and B_2 both are non-empty subsets of B, therefore $B_1 \cap B_2$ is also a non-empty subset of B.

Suppose $a, b \in B_1 \cap B_2$, which implies that

$a, b \in B_1$ and $a, b \in B_2$ …(1)

$\Rightarrow$ $a + b \in B_1$ and $a + b \in B_2$ [as B_1 and B_2 are ideals of B]

$\Rightarrow$ $a + b \in B_1 \cap B_2$ …(2)

Again as $a \in B_1 \cap B_2$

$\Rightarrow$ $a \in B_1$ and $a \in B_2$

If $s \in B$, then

$a \bullet s \in B_1$ and $a \bullet s \in B_2$

$\Rightarrow$ $a \bullet s \in B_1 \cap B_2$ …(3)

Thus we have proved that

(i) $a, b \in B_1 \cap B_2 \Rightarrow a + b \in B_1 \cap B_2$ from (2)

and, (ii) $a \in B_1 \cap B_2$ $s \in B \Rightarrow a \bullet s \in B_1 \cap B_2$ from (3)

Hence, $B_1 \cap B_2$ is an Ideal of B.

Theorem 6: Union of two idea is an ideal if and only if one of them is contained in the other.

Proof: Suppose that B_1 and B_2 are two ideals of a Boolean algebra B. Also suppose that $B_2 \subseteq B_1$. We have to prove that $B_1 \cup B_2$ is an ideal of B.

(i) First we shall prove that the condition is necessary

i.e. if $B_2 \subseteq B_1$, then $B_1 \cup B_2$ be an ideal of B

As $B_2 \subseteq B_1$, we have $B_1 \cup B_2 = B_1$ which is an ideal of B.

Therefore, $B_1 \cup B_2$ is also an ideal of B.

(ii) Now we shall prove that the condition is sufficient B.

i.e. if $B_1 \cup B_2$ is an ideal of B; then either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$. We shall use method of contradition.

Let $B_1 \cup B_2$ be an ideal of B. Also suppose that $B_1 \not\subseteq B_2$ and $B_2 \not\subseteq B_1$.

$\Rightarrow$ there exists an element $a \in B_1$ such that $a \notin B_2$ …(1)

and also there exists an element $b \in B_2$ such that $b \notin B_1$ …(2)

$\Rightarrow$ $a, b \in B_1 \cup B_2$

$\Rightarrow$ $a + b \in B_1 \cup B_2$ (as $B_1 \cup B_2$ is an ideal of B)

$\Rightarrow$ $(a + b) \in B_1$ or $(a + b) \in B_2$ …(3)

Now if $(a + b) \in B_1$ and $b \in B_2$, then $(a + b) \bullet b \in B_1$ (as B_1 is an ideal of B)

$(a + b) \bullet b \in B_1 \Rightarrow b \in B_1$ (by absorption law)

which contradicts our assumption in (2).

Similarly if we suppose another possibility of (3) i.e. $a + b \in B_2$ we shall get $a \in B_2$ which contradicts (1).

Hence our assumptions that $B_1 \not\subseteq B_2$ and $B_2 \not\subseteq B_1$ are not valid. If means one of these is contained in the other.

Theorem 7: The necessary and sufficient codition for a non-empty subset B′ of Boolean algebra to be an ideal of B is

(i) $a, b \in B' \Rightarrow (a + b) \in B'$

(ii) $a \in B', x \le a \Rightarrow x \in B'$

Proof: *Condition is necessary:* Let B′ be an ideal of B

then $a, b \in B' \Rightarrow (a + b) \in B'$ (by definition) which proves part (i)

Again if $a \in B', x \le a$, then $x = a \bullet x \in B'$ (B' being ideal) which proves part (ii)

Condition is sufficient: Let $a, b \in B' \Rightarrow (a + b) \in B'$

and $a \in B', x \le a \Rightarrow x \in B'$

We have to prove that B' is an ideal of B for which

We have to show only that

$$a \in B', s \in B \Rightarrow a \bullet s \in B'$$

As $a \bullet s \le a$ and $a \in B'$

$$a \bullet s \in B'$$

Hence, B' is an ideal of B.

6. DIRECT PRODUCT OF BOOLEAN ALGEBRAS

If $(B_1, +, \bullet, ' \, 0_1, 1)$ and $(B_2, +_2, \bullet_2, '', 0_2, 1)$ are two Boolean algebras, then their direct product is defined as a Boolean algebra given by

$$(B_1 \times B_2, +_3, \bullet_3, ''', 0_3, 1_3)$$

where the operations $+_3, \bullet_3$ and $'''$ are defined for any (a_1, b_1) and $(a_2, b_2) \in B_1 \times B_2$ as follows:

$$(a_1, b_1) +_3 (a_2, b_2) = (a_1 +_1 a_2, b_1 +_2 b_2)$$

$$(a_1, b_1) \bullet_3 (a_2, b_2) = (a_1 \bullet_1 a_2, b_1 \bullet_2 b_2)$$

$$(a_1, b_1)''' = (a_1', b_1'')$$

$$0_3 = (0_1, 0_2) ; 1_3 = (1_1, 1_2)$$

7. BOOLEAN HOMOMORPHISM AND ISOMORPHISM

Let $(B_1, +, \bullet, ' \, 0, 1)$ and $(B_2, *, \oplus, -, \alpha, \beta)$ be two Boolean algebras. A mapping defined as $f : B_1 \to B_2$ is called a Boolean homomorphism if all the operations of the Boolean algebra are preserved. It means that for any

$$a, b \in B$$

$$f(a + b) = f(a) * f(b)$$

$$f(a \bullet b) = f(a) * f(b)$$

$$f(a') = f(\bar{a})$$

$$f(0) = \alpha$$

$$f(1) = \beta$$

If the mapping f is one-one also in addition to being homomorphic, then this mapping is called Isomorphism.

In particular if B_1 and B_2 are two Boolean algebraic with respect to the same operations +, $\bullet$, 0 and 1 then $f : B_1 \to B_2$ is called isomorphism if

(i) f is one-one

(ii) $f(a + b) = f(a) + f(b)$

(iii) $f(a \bullet b) = f(a) \bullet f(b)$

(iv) $f(a') = f(a)'$

for any $a, b \in B_1$ and B_1 and B_2 are said to be isomorphic.

Stone's representation theorem: It states that any Boolean algebra is isomorphic to a power set algebra $\left(P(S), \cap, \cup, \sim, \phi, S\right)$ for some set S.

Theorems on Boolean Algebra

Theorem 8: For any Boolean algebra (B, +, • , /)

(i) identity for the operation + is unique.

(ii) identity for the operation • is unique.

(iii) For each $a \in B$, the complement of a is also unique.

Proof: **(i)** If possible, let 0_1 and 0_2 be the two identities for the operation +, $(0_1, 0_2 \in B)$ then $0_1 \in B$ is the identity and $0_2 \in B$, we have

$$0_2 + 0_1 = 0_2 = 0_1 + 0_2 \qquad \ldots(1)$$

Similarly $0_2 \in B$ is the identity and $0_1 \in B$, we have

$$0_1 + 0_2 = 0_1 = 0_2 + 0_1 \qquad \ldots(2)$$

Therefore from (1) and (2), we have $0_2 = 0_1$

Hence identity for the + operation is unique.

Part (ii) can be proved in a similar way.

(iii) If possible let b and c be two complements of $a \in B$, $(b, c \in B)$, then

$b = b + 0$ (0 being identity for +)

$= b + a \bullet c$ (since c is complement of a)

$= (b + a) \bullet (b + c)$ (by distributive law)

$= (a + b) \bullet (b + c)$ (by commutative law)

$= 1 \bullet (b + c)$ (since b is complement of a)

$= b + c$ (1 is identity of b + c for •) …(1)

Similarly $c = c + 0 = c + a \bullet b$ (b is complement of a)

$= (c + a) \bullet (c + b)$ (by distributive law)

$= (a + c)(b + c)$ (by commutative law)

$= 1 \bullet (b + c)$ (since c is complement of a)

$= b + c$ …(2)

From (1) and (2) we have $b = c$

Thus the complement of a is unique.

Theorem 9: **Idempotent laws:** If $a \in B$, then for any Boolean algebra (B, +, •, /) or $(B, \vee, \wedge, /)$

(i) $a \vee a = a$ or $a + a = a$

(ii) $a \wedge a = a$ or $a \bullet a = a$

Note: We may use any one of the two set of operations i.e. $\vee$ and $\wedge$ or + and •.

Solution: (i) $a + a = (a + a) \bullet 1$ (1 is identity for •)

$= (a + a) \bullet (a + a')$ (a′ is complement of a)

$= a + a \bullet a'$ (by distributive law)

$= a + 0$ (Since $a \bullet a' = 0$, 0 being identity for +)

$= a$

(ii) $a \bullet a = a \bullet a + 0$ (0 is identity for +)

$= a \bullet a + a \bullet a'$ (a′ is complement of a)

$= a \bullet (a + a')$ (by distributive law)

$= a \bullet 1$ (1 is identity for •)

$= a$

Theorem 10: Boundedness law or Null law: For any Boolean algebra (B, +, •, /) or (B, ∨, ∧, , /)

(i) $a + 1 = 1$ or $a \vee I = I$ (ii) $a \bullet 0 = 0$ or $a \wedge 0 = 0 \ \forall \ a \in B$.

Proof: (i) $a + 1 = 1 \bullet (a + 1)$ (1 is identity for •)

$= (a + a') \bullet (a + 1)$ (a′ is complement of a)

$= a + a' \bullet 1$ (by distributive law)

$= a + a'$ (as $a' \bullet 1 = a'$)

$= 1$

(ii) Similarly

$a \bullet 0 = 0 + a \bullet 0$

$= a \bullet a' + a \bullet 0$

$= a \bullet (a' + 0)$

$= a \bullet a'$

$= 0$

Theorem 11: Absorption law: For any Boolean algebra (B, +, •, /) or (B, ∨, ∧, /),

(i) $a + (a \bullet b) = a$ or $a \vee (a \wedge b) = a$

(ii) $a \bullet (a + b) = a$, or $a \wedge (a \vee b) = a \ \forall \ a, b \in B$.

Proof: (i) Let $a, b \in B$, then

$a + a \bullet b = a \bullet 1 + a \bullet b$ (1 is identity for •)

$= a \bullet (1 + b)$

$= a \bullet (b + 1)$ (by commutative law)

$= a \bullet 1$ (since $b + 1 = 1$ by boundedness law)

$= a$

(ii) $a \bullet (a + b) = (a + 0) \bullet (a + b)$ (0 is identity for +)

$= a + (0 \bullet b)$ (by distributive law)

$= a + b \bullet 0$ (by commutative law)

$= a + 0$ (as $b \bullet 0 = 0$)

$= a$

Theorem 12: Involution law: For any Boolean algebra (B, +, •, /) or (B, ∨, ∧, /), $\forall \ a \in B$,

$(a')' = a$

Proof: $(a')' = 1 \bullet (a')'$ (1 is identity for •)

$= (a + a') \bullet (a')'$ (a′ is complement of a)

$= a \bullet (a')' + a' \bullet (a')'$ (by distributive law)

$= a \bullet (a')' + 0$ (as (a′)′ is complement of a′)

$= 0 + a \bullet (a')'$ (by commutative law)

$= a \bullet a' + a \bullet (a')'$

$= a \bullet (a' + (a')')$ (by distributive law)

$= a \bullet 1$ (as $a' + (a')' = 1$)

$= a$.

Theorem 13: De-Morgan's laws: For a Boolean algebra (B, +, •, /) or (B, ∨, ∧, , /) we have

(i) $(a + b)' = a' \bullet b'$ or $(a \vee b)' = a' \wedge b'$ **[M.Sc. (Math) 2004]**

(ii) $(a \bullet b)' = a' + b'$ or $(a \wedge b)' = a' \vee b' \ \forall \ a, b \in B$. **[M.Sc. (Math) 2004-05]**

Proof: (i) We have to prove that the complement of $a + b$ is $a' \bullet b'$ for which we shall have to prove that

$$(a + b) + a' \bullet b' = 1 \quad \dots(1)$$

and

$$(a + b) \bullet (a' \bullet b') = 0 \quad \dots(2)$$

where 1 and 0 are identities for the operation • and + respectively.

To prove (1), we have its L.H.S.

$= (a + b) + a' \bullet b' = [(a + b) + a')] \bullet [(a + b) + b']$ (by distributive law)

$= [(a + a') + b] \bullet [a + (b + b')]$ (by associative and communicative laws)

$= [1 + b] \bullet [a + 1]$

$= 1 \bullet 1$ [as $a + 1 = b + 1 = 1$ by theorem 10]

$= 1.$

To prove (2) we have its L.H.S

$= (a + b) \bullet (a' \bullet b') = a\bullet(a'\bullet b') + b\bullet(a'\bullet b')$ (by distributive law)

$= (a\bullet a')\bullet b' + a'\bullet(b\bullet b')$ (by associative and commutative laws)

$= 0\bullet b' + a'\bullet 0$ (as $a\bullet a' = 0$, $b\bullet b' = 0$)

$= b'\bullet 0 + a'\bullet 0$

$= 0 + 0$ (as $a'\bullet 0 = b'\bullet 0 = 0$)

$= 0.$

Thus having proved (1) and (2) it is proved that $(a + b)' = a'\bullet b'$.

(ii) Here we have to prove that the complement of $(a\bullet b)$ is $a' + b'$ for which we shall have to prove that

$(a\bullet b) + (a' + b') = 1$ …(3)

and $(a\bullet b)\bullet(a' + b') = 0$ …(4)

To prove (3) its L.H.S.

$= (a \bullet b) + (a' + b') = [a + (a' + b')]\bullet[b + (a' + b')]$ (by distributive law)

$= [(a + a') + b'] \bullet [(b + b') + a')]$ (by commutative and associative laws)

$= (1 + b')(1 + a')$ [as $a + a' = 1$, $b + b' = 1$]

$= (b' + 1)\bullet(a' + 1)$

$= 1 \bullet 1$

$= 1.$

To prove (4) we have its L.H.S.

$= (a\bullet b)\bullet(a' + b') = (a\bullet b\bullet a') + (a\bullet b)\bullet b'$ (by distributive law)

$= (a\bullet a')\bullet b + a\bullet(b\bullet b')$ (by associative and commutative laws)

$= 0\bullet b + a\bullet 0$ (as $a\bullet a' = 0$, $b\bullet b' = 0$)

$= b\bullet 0 + a\bullet 0$ (0 is identitity for +)

$= 0 + 0$ (as $b\bullet 0 = 0$, $a\bullet 0 = 0$)

$= 0$

Having proved (3) and (4) we have proved that $(a\bullet b)' = a' + b'$.

Theorem 7: In a Boolean algebra $(B, +, \bullet, /)$

(i) $0' = 1$ and (ii) $1' = 0$

where 0 and 1 are the identities for + and • operations respectively.

Proof: (i) $0' = 0 + 0'$

$= 1$ (as $a + a' = 1 \ \forall \ a \in B$)

(ii) $1' = 1\bullet 1'$

$= 0$ (as $a\bullet a' = 0 \ \forall \ a \in B$)

8. APPLICATION OF PREVIOUS THEOREMS IN SOLVING PROBLEMS

Example 3: If (B, +, •, /) is a Boolean algebra and a, b ∈ B then prove that

(i) $a + a' b = a + b$ (ii) $a \bullet b = a \Rightarrow a \bullet b' = 0$

Solution: (i) $a + b = (a + b) \bullet 1$

$= (a + b) \bullet (a + a')$ (as $1 = a + a'$)

$= a + b \bullet a'$ (by distributive law)

$= a + a' \bullet b$ (by commutative law)

or $a + a' \bullet b = a + b$

(ii) $a \bullet b' = (a \bullet b) b'$ (as $a = a \bullet b$ is given)

$= a \bullet (b \bullet b')$ (by associative law)

$= a \bullet 0$ (as $b \bullet b' = 0$)

$= 0$ (by theorem 10)

Example 3: Prove that in a Boolean algebra (B, +, •, /), $a \bullet b + b \bullet c + c \bullet a = (a + b) \bullet (b + c) \bullet (c + a)$ $\forall$ a, b, c ∈ B.

Proof: R.H.S. $= (a + b) \bullet (b + c) \bullet (c + a)$

$= (a + b) \bullet [(c + b) \bullet (c + a)]$ (by commutativity)

$= (a + b) \bullet [c + b \bullet a]$ (by distributivity)

$= a \bullet c + b \bullet c + a \bullet (b \bullet a) + b \bullet (b \bullet a)$ (by distributivity)

$= a \bullet c + b \bullet c + (a \bullet a) \bullet b + (b \bullet b) \bullet a$ (by commutativity and associativity)

$= a \bullet c + b \bullet c + a \bullet b + b \bullet a$ (as $a \bullet a = a$, $b \bullet b = b$)

$= a \bullet c + b \bullet c + (a \bullet b + a \bullet b)$

$= a \bullet c + b \bullet c + a \bullet b$ (as $a + a = a$)

$= a \bullet b + b \bullet c + c \bullet a$

Example 4: Prove that: (i) $(a + b) = b \Leftrightarrow a' + b = 1$ (ii) $a' + b = 1 \Leftrightarrow a \bullet b' = 0$

Solution: (i) To prove it we shall prove

(a) $a + b = b \Rightarrow a' + b = 1$

and (b) $a' + b = 1 \Rightarrow a + b = b$

To prove (a), let $a + b = b$ then, $a' + b = a' + (a + b)$

$= (a' + a) + b = 1 + b = 1$

To prove (b) let $a' + b = 1$, then $a + b = 1 \bullet (a + b)$

$= (a' + b) \bullet (a + b)$

$= (a' \bullet a) + b$

$= 0 + b = b$

Hence from (a) and (b), we have

$(a + b) = b \Leftrightarrow a' + b = 1$

(ii) To prove it we shall have to prove that

(a) if $a' + b = 1$, then $a \bullet b' = 0$ or $a' + b = 1 \Rightarrow a \bullet b' = 0$

and (b) if $a \bullet b' = 0$ then $a' + b = 1$ or $a \bullet b' = 0 \Rightarrow a' + b = 1$

To prove (a), Let $a' + b = 1$, then $a \bullet b' = (a')' b'$

$= (a' + b)'$ (by De, Morgan law)

$= (1)'$ [$a' + b = 1$ is given]

$= 0$ [Complement of 1 = 0]

To prove (b), let $a \bullet b' = 0$, then $a' + b = a' + (b')' = 0$

$$= (a \bullet b')'$$
$$= (0)'$$
$$= 1$$

From (a) and (b) we have

$$a' + b = 1 \Leftrightarrow a \bullet b' = 0$$

Example 5: If $(B, +, \bullet, 0, 1, ')$ is a Boolean algebra and $a, b \in B$, then prove that

(i) $a + (a + b) = a + b$ (ii) $a \bullet (a \bullet b) = a \bullet b$

(iii) $a + a' \bullet b = a + b$ (iv) $a' + a \bullet b = a' + b$

Solution: (i) $a + (a + b) = (a + a) + b$ (associative law)

$= a + b$ (as $a + a = a$)

Hence proved.

(ii) $a \bullet (a \bullet b) = (a \bullet a) \bullet b$ (associative law)

$= a \bullet b$ (as $a \bullet a = a$)

Hence proved.

(iii) $a + a' \bullet b = (a + a') \bullet (a + b)$ (by distributive law)

$= 1 \bullet (a + b)$

$= a + b$

Hence proved.

(iv) $a' + a \bullet b = (a' + a) \bullet (a' + b)$ (by distributive law)

$= 1 \bullet (a' + b)$

$= a' + b$

Hence, proved.

Example 6: Let (B, +, *, /) be a Boolean algebra and $a, b, x \in B$. Then,

(i) if $a * x = b * x$ and $a * x' = b * x'$, prove that $a = b$

(ii) if $a + x = b + x$ and $a + x' = b + x'$, prove that $a = b$

(iii) if $x + a = x + b$ and $x * a = x * b$, prove that $a = b$. **[M.Sc. (Maths) 2004]**

Solution: (i) $a * x = b * x$ …(i)

$a * x' = b * x'$ …(2)

Combining the elements on LHS and RHS, of (1) and (2) by the operation +, we have

$a * x + a * x' = b * x + b * x'$

or $a * (x + x') = b * (x + x')$ (distributive law)

or $a * 1 = b * 1$

or $a = b$

(ii) $a + x = b + x$ …(1)

$a + x' = b + x'$ …(2)

Combining the elements on LHS and RHS of (1) and (2) by the operation * we have

$(a + x) * (a + x') = (b + x) * (b + x')$

or $a + x * x' = b + x * x'$ (by distributive law) or $a + 0 = b + 0$ (as $x * x' = 0$)

or $a = b$

(iii) $a = a + a \bullet x$ (by absorption law)

$= a \bullet a + a \bullet x$ (as $a \bullet a = a$)

$= a \bullet (a + x)$ (by distributive law)

$= a \bullet (x + a)$ (by commutative law)

$= a \bullet (x + b)$ ($x + a = x + b$ is given)

$= a \bullet x + a \bullet b$ (by distributive law)

$= x \bullet a + a \bullet b$ (by commutative law)

$= x \bullet b + a \bullet b$ (as $x \bullet a = x \bullet b$ is given)

$= (x + a) \bullet b$ (by distributive law)

$= (x + b) \bullet b$ ($x + a = x + b$ is given)

$= x \bullet b + b \bullet b$

$= x \bullet b + b$

$= b + b \bullet x$ (by commutiative law)

$= b$ (by absorption law)

Example 7: Let $(B, +, \bullet, /)$ be a Boolean algebra with 0 and 1 as identities for the operations + and $\bullet$ respectively and $a, b \in B$. Prove that:

(i) $a \bullet b = a \Rightarrow a \bullet b' = 0$ (ii) $a \bullet b' = 0 \Rightarrow a + b = b$ (iii) $a + b = b \Rightarrow a \bullet b = a$

Solution: (i) $a \bullet b = a$ (given)

Now, $a \bullet b' = (a \bullet b) \bullet b'$ (Putting $a = a \bullet b$ as given)

$= a \bullet (b \bullet b')$ (by associative law)

$= a \bullet 0$ (as $b \bullet b' = 0$)

$= 0$ (as $a \bullet 0 = 0$)

(ii) $a \bullet b' = 0$ (given)

Now, $a + b = (a + b) \bullet 1$ (Identity property)

$= (a + b) \bullet (b + b')$

$= (b + a) \bullet (b + b')$ (commutative law)

$= b + a \bullet b'$ (by distributive law)

$= b + 0$ (as $a \bullet b' = 0$ is given)

$= b$

Hence proved.

(iii) $a + b = b$ (given)

Now, $a \bullet b = a \bullet (a + b)$ (as replacing b by a + b as given)

$= a \bullet a + a \bullet b$

$= a + a \bullet b$

$= a$ (by absorption law)

Hence proved.

Example 8: In Boolean algebra (B, +, •, /), prove that $a + b = 0 \Leftrightarrow a = 0, b = 0$.

Solution: Let a = 0 and b = 0, then

$$a + b = 0 + 0 = 0 \qquad \ldots(1)$$

Again let a + b = 0, then

$$a = a + 0 = a + b \bullet b' = (a + b) \bullet (a + b')$$
$$= 0 \bullet (a + b') \quad \text{as } (a + b = 0 \text{ given})$$
$$= 0$$

and $b = b + 0 = b + a \bullet a' = (b + a) \bullet (b + a')$

$$= (a + b) \bullet (b + a') = 0 \bullet (b + a') = 0 \qquad \ldots(2)$$

From (1) and (2) the result is proved.

Example 9: If $(B, +, \bullet, 0, 1, /)$ is a Boolean algebra and $a, b, c \in B$ then. Prove the following:

(i) $(a + b) \bullet (a' + c) = a' \bullet b + a \bullet c$

(ii) $a \bullet b \bullet c + a \bullet b \bullet c' + a \bullet b' \bullet c + a' \bullet b \bullet c = a \bullet b + b \bullet c + c \bullet a$

Solution: (i) LHS $= (a + b) \bullet (a' + c)$

$= (a + b) \bullet a' + (a + b) \bullet c$ (by distributive law)

$= a \bullet a' + b \bullet a' + a \bullet c + b \bullet c$ (by distributive law)

$= 0 + b \bullet a' + a \bullet c + b \bullet c$ (as $a \bullet a' = 0$)

$= a \bullet c + b \bullet a' + b \bullet c$ (by associative law)

$= a \bullet c + b \bullet a' + bc \bullet 1$

$= a \bullet c + b \bullet a' + b \bullet c \bullet (a + a')$

$= a \bullet c + b \bullet a' + b \bullet c \bullet a + b \bullet c \bullet a'$

$= a \bullet c + b \bullet c \bullet a + a' b + a' \bullet b \bullet c$ (by associative and commutative law)

$= a \bullet c + a \bullet c \bullet b + a' \bullet b \bullet (1 + c)$

$= a \bullet c \bullet (1 + b) + a' \bullet b \bullet 1$ (as $1 + c = 1$)

$= a \bullet c \bullet 1 + a' \bullet b$

$= a \bullet c + a' \bullet b$

$= a' b + a \bullet c$

(ii) LHS $= a \bullet b \bullet c + a \bullet b \bullet c' + a \bullet b' \bullet c + a' \bullet b \bullet c$

$= a \bullet b\,(c + c') + a \bullet b' \bullet c + a' \bullet b \bullet c$

$= a \bullet b \bullet 1 + a \bullet b' \bullet c + a' \bullet b \bullet c$ (as $c + c' = 1$)

$= a \bullet b + a \bullet b' \bullet c + a' \bullet b \bullet c$

$= a\,[b + b' \bullet c] + a' \bullet b \bullet c$ (by distributive law)

$= a \bullet [(b + b') \bullet (b + c)] + a' \bullet b \bullet c$ (by distributive law)

$= a \bullet [1 \bullet (b + c)] + a' \bullet b \bullet c$ (as $b + b' = 1$)

$= a \bullet (b + c) + a' \bullet b \bullet c$

$= a \bullet b + a \bullet c + a' \bullet b \bullet c$

$= a \bullet b + (a + a' \bullet b) \bullet c$ (by distributive law)

$= a \bullet b + (a + a') \bullet (a + b) \bullet c$ (by distributive law)

$= a \bullet b + 1 \bullet (a + b) \bullet c$ (as $a + a' = 1$)

$= a \bullet b + a \bullet c + b \bullet c$ (by distributive law)

$= a \bullet b + b \bullet c + c \bullet a$ (by associative and commutative law)

Example 10: Let $(B, +, \bullet, /, 0, 1)$ be a Boolean algebra and $x, y, z \in B$. Prove that:

(i) $(x + y) \bullet (x' + z) \bullet (y + z) = x \bullet z + y \bullet z + x'y$

(ii) $(x + y) \bullet (x' + z) = x \bullet z + y \bullet z + x'y$

(iii) $x \bullet z + y \bullet z + x' \bullet y = x \bullet z + x' \bullet y$

Solution:

(i) $(x + y) \bullet (x' + z) \bullet (y + z)$

$= (x + y) \bullet (z + x') \bullet (z + y)$ (by commutative law)

$= (x + y) \bullet (z + x' \bullet y)$ (by distributive law)

$= x \bullet z + x \bullet x' \bullet y + y \bullet z + y \bullet x' \bullet y$ (by distributive law)

$= x \bullet z + 0 \bullet y + y \bullet z + x' \bullet y \bullet y$ (as $x \bullet x') = 0$

$= x \bullet z + y \bullet z + x' \bullet y$ (as $y \bullet y = y$)

Hence proved.

(ii) $(x + y) \bullet (x' + z)$

$= x \bullet x' + x \bullet z + y \bullet x' + y \bullet z$

$= 0 + x \bullet z + y \bullet z + x' \bullet y = xz + yz + x' \bullet y$ (by associative and commutative laws)

Hence proved.

(iii) $x \bullet z + y \bullet z + x' \bullet y$

$= xz + x' \bullet y + y \bullet z \bullet 1$

$= x \bullet z + x' \bullet y + y \bullet z\,(x + x')$ (as $x + x' = 1$)

$= x \bullet z + x' \bullet y + y \bullet z \bullet x + y \bullet z \bullet x'$

$= x \bullet z + x \bullet yz + x'y + x' \bullet y \bullet z$ (by associative and commutative laws)

$= (x \bullet z + x \bullet z \bullet y) + (x' \bullet y + x' \bullet y \bullet z)$ (by commutative law)

$= x \bullet z + x'y$ (by absorption law)

Example 11: Express each of the following propositional statements in Boolean algebra $(B, +, \bullet /, 0, 1)$ and then simplify $(p, q \in B)$.

(i) $p \Rightarrow q$ (ii) $p \Leftrightarrow q$ (iii) $(p \wedge q) \vee \neg[(p \vee (\neg q)) \wedge q]$

(symbols $\wedge$ and $\vee$ may be taken as $\bullet$ and + for the sake of convenience)

Solution: (i) $p \Rightarrow q = (\neg p \vee q) = p' + q$

(ii) $p \Leftrightarrow q = (p \Rightarrow q) \wedge (q \Rightarrow p)$

$= (\neg p \vee q) \wedge (\neg q \vee p)$

$= (p' + q) \bullet (q' + p)$

$= p' \bullet q' + p' \bullet p + q \bullet q' + q \bullet p$ (by distributive law)

$= p' \bullet q' + p \bullet p' + q \bullet q' + p \bullet q$ (by commutative law)

$= p' \bullet q' + 0 + 0 + p \bullet q$ (as $p \bullet p' = q \bullet q' = 0$)

$= p \bullet q + p' \bullet q'$

(iii) $(p \wedge q) \vee \neg[(p \vee (\neg q)) \wedge q]$

$= p \bullet q + [(p + q') \bullet q]'$

$= p \bullet q + (p + q')' + q'$ (by DeMorgan's law)

$= p \bullet q + p' \bullet q + q'$

$= (p + p') \bullet q + q'$ (by distributive law)

$= 1 \bullet q + q'$ (as $p + p' = 1$)

$= q + q'$

$= 1$ (as $q + q' = 1$)

9. DUALITY

Definition: The dual of any statement in a Boolean algebra B is the statement which is obtained by interchanging the operation + and • (or $\vee$ and $\wedge$) and also interchanging their identity elements 0 and 1 in the original statement.

For example the dual of $a + 1 = 1$ is $a \bullet 0 = 0$, the dual of $(0 \bullet a) + (b \bullet 1) = b$ is $(1 + a) \bullet (b + 0) = b$.

Principle of Duality: The dual of any theorem in a Boolean algebra is also a theorem.

10. BOOLEAN EXPRESSIONS

Let (B, +, •, /) or (B, $\vee$, $\wedge$, /) be a Boolean algebra, where $B = \{x_1, x_2 \ldots\}$ is a non empty set, + or $\vee$ and • or $\wedge$ are two binary operations, / is a unary operation. 0 is the identity element for the operation + or $\vee$ and 1 is the identity for the operation • or $\wedge$. Then $x_1, x_2, \ldots$ are called variables. A variable x_i can assume the value x_i or its complemented value x_i'. 0 and I also belong to B.

10.1 Definition

1. Literal: A literal is a variable or a complemented variable such as x, x′, y, y′ & so on x_2, x_2' are two literals involving one variable x_2.

2. Boolean Expression: Let B = (X, $\wedge$, $\vee$, ′, 0, I) or $(X, \bullet +, ', 0; 1)$ be a Boolean algebra. A Boolean expression in variables $x_1, x_2 \ldots x_k$ each taking their values in the set X is defined recursively as follows:

(1) Each of the variables $x_1, x_2, \ldots x_k$ as well as the elements 0 and I of B are Boolean expressions.

(2) If X_1 and X_2 are previously defined Boolean expressions, then $X_1 \wedge X_2$, $X_1 \vee X_2$ and X_1' are also Boolean expressions. e.g. x_1, x_3' are Boolean expressions. By (1) and (2)

$x_1 \wedge x_3'$ is also Boolean expressions by (2)

$x_1 \vee x_3'$ is also Boolean expressions by (2)

$(x_1 \vee x_2) \wedge (x_1 \wedge x_3')$ is also Boolean expressions by (2) repeatedly

A Boolean expression in $x_1, x_2, \ldots x_n$ is denoted as $X = X(x_1, x_2, \ldots x_n)$.

Similarly $0 + x_1', (x_1 \bullet x_2)', (x_1 + x_2) \bullet (x_1 \bullet x_2)'$ are Boolean expressions.

In fact a Boolean expression generated by $x_1, x_2, \ldots, x_k$ is a combination of elements of B and the operations of meet, join and complementation.

3. Minterms: A *fundamental product* or a *minterm* is a literal or a product or meet of two or more literals in which no two literals involve the same variable.

Thus xz′, xy′z, x, y′, x′yz are fundamental products, (these can be written as $x \wedge z'$, $x \wedge y' \wedge z$, x, y′, $x' \wedge y \wedge z$ also) but xyx′z′ and xyzy′ are not literals as x, x′ and y, y′ involve the same variables x and y respectively.

Any product of literals can be reduced to either 0 or a fundamental product

e.g. $x \bullet y \bullet x' \bullet z = 0$ since $x \bullet x' = 0$ (complement law)

and $x \bullet y \bullet z \bullet y = x \bullet y \bullet z$ (idempotent law)

A fundamental product P_1 is said to be contained in or included in another fundamental product P_2, if the literals of P_1 are also literals of P_2

e.g. $x'z$ is contained in $x'yz$

But $x'z$ is not contained in $xy'z$ since x' is not a literal of $xy'z$.

P_1 is contained in P_2 means $P_2 = P_1 \bullet Q$, then by the absorption law $P_1 + P_2 = P_1 + P_1 \bullet Q = P_1$

Thus for instance $x' \bullet z + x' \bullet y \bullet z = x' \bullet z$.

(Here + and • are two binary operation of the Boolean algebra).

Alternative Definition of Fundamental product or Minterm

A Boolean expression in k variables $x_1, x_2, \ldots x_k$ is called a **minterm** if it is of the form $y_1 \wedge y_2 \wedge \ldots \wedge y_k$. where each y_j is a literal (i.e. either x_i or x'_i) for $1 \le i \le k$ and $y_i \ne y_j$ for $i \ne j$.

It means a minterm in k variables is a product or meet of exactly k distinct variables e.g. $x_1 \wedge x_2'$ is a minterm in two variables x_1 and x_2.

Note: $x_1 \wedge x_2'$ means the same thing as $x_1 \bullet x_2'$.

Note: Distinct variables means literals none of which involves the same variables i.e. literals x_i & x_i' are not distinct as both involves the same variable x_i.

11. SUM OF PRODUCTS EXPRESSION

Definition: A Boolean expression E is called a sum-of-products expression if E is a fundamental product or the sum of two or more fundamental products none of which is contained in another.

[**Note:** Such type of fundamental products are called distinct minterm].

Definition: Let E be any Boolean expression. A sum-of-products form of E is an equivalent Boolean sum-of-products expression.

Illustration: Consider $E_1 = xz' + y'z + xyz'$

and $E_2 = xz' + x'yz' + xy'z$

Although E_1 is a sum of products it is not a sum-of-products expression, as product xz' is contained in the product xyz'. However by absorption law, E_1 can be expressed as

$$E_1 = xz' + y'z + xyz' = xz' + xyz' + y'z$$
$$= xz' + y'z$$

This gives a sum-of-product expression form for $E_1 \bullet E_2$ is already a sum-of-product expression.

Example 12: Express $E = ((xy)'z)' \, ((x' + z)(y' + z'))'$ as a sum of product expression

Solution: $E = ((xy)'' + z') \, ((x' + z)' + (y' + z')')$ (by De-Morgan's laws)

$= (xy + z') \, (x \bullet z' + yz)$

$= xyxz' + xyyz + xz'z' + yzz'$ (by distributive law)

$= xyz' + xyz + xz' + 0$ By commutative, idempotent & complement laws.

[each term is a fundamental product or zero, but xz' is contained in xyz']

$= xyz + xz' + xyz' = xyz + xz'$ (By absorption law and identity law)

[a sum of product expression]

Alternative definition of sum of product expression: Sum of product expression is a sum (or join) of distinct minterms (i.e. fundamental products none of which is contained in the other.)

12. DISJUNCTIVE NORMAL FORM (DNF) OR COMPLETE SUM-OF-PRODUCT FORMS OR DISJUNCTIVE CANONICAL FORM

Definition: A Boolean expression $E = (x_1, x_2, \ldots x_n)$ is said to be a complete sum of products expression if E is a sum-of-products expression where each product P involves all the n variables. Such a fundamental product P which involves all the variables is called a minterm and there is a maximum of 2^n such products for n variables.

Illustration: $x \bullet (y' \bullet z)'$ or $x \wedge (y' \wedge z)'$

$= x \bullet (y + z')$

$= x \bullet y + x \bullet z'$ (sum of product expression)

$= x \bullet y \bullet (z + z') + x \bullet z' \bullet (y + y')$

$= x \bullet y \bullet z + x \bullet y \bullet z' + x \bullet y \bullet z' + x \bullet y' \bullet z'$

$= x \bullet y \bullet z + x \bullet y \bullet z' + x \bullet y' \bullet z'$ (complete sum-of-products form or DNF)

The last result can also be written as

$$(x \wedge y \wedge z) \vee (x \wedge y \wedge z') \vee (x \wedge y' \wedge z')$$

Alternative definition of complete sum of product expression or Disjunctive Normal form (DNF)

A Boolean expression involving k variables is in disjinctive normal form (DNF) if it is a join or sum of distinct minterms each involving exactly k variables e.g. the Boolean expression in 2 variables.

$$X(x_1, x_2) = (x_1' \wedge x_2') \vee (x_1 \wedge x_2') \vee (x_1' \wedge x_2) \quad \ldots(1)$$

is in DNF, as it a join of 3 distinct minterms each involving exactly 2 variables.

Note: Distinct minterms means that none of the minterm is contained in the other.

The expression (1) in the notation of + and • can be written as $X(x_1, x_2) = x_1' \bullet x_2 + x_1 \bullet x_2' + x_1' \bullet x_2$

13. COMPLETE DISJUNCTIVE NORMAL FORM

A DNF in n variables which contains all the possible 2^n terms is called the complete DNF in n variables. For example a complete DNF in two variables is $xy + x'y + xy' + x'y'$ (it contains $2^2 = 4$ terms) and complete DNF in three variables is $xyz + xyz' + xy'z + x'yz + x'y'z + xy'z' + x'yz' + x'y'z'$ (it contains $2^3 = 8$ terms).

Note: A complete D.N.F. is identitically 1.

Example 13: Obtain a disjunctive normal form for the expression

$$X(x_1, x_2, x_3) = (x_1' \wedge x_2) \vee (x_1 \wedge x_3) \quad \ldots(1)$$

Solution: $x_1' \wedge x_2 = (x_1' \wedge x_2) \wedge I$ (Identity law)

$= (x_1' \wedge x_2) \wedge (x_3 \vee x_3')$ (complementation law)

$= (x_1' \wedge x_2 \wedge x_3) \vee (x_1' \wedge x_2 \wedge x_3')$ (Distributive law) …(2)

Also $x_1 \wedge x_3 = (x_1 \wedge x_3) \wedge I$ (Identity law)

$= (x_1 \wedge x_3) \wedge (x_2 \vee x_2')$ (complementation law)

$= (x_1 \wedge x_3 \wedge x_2) \vee (x_1 \wedge x_3 \wedge x_2')$ (distributive law)

$= (x_1 \wedge x_2 \wedge x_3) \vee (x_1 \wedge x_2' \wedge x_3)$ (commutativity law) …(3)

Therefore putting values from (2) and (3) we have from (1)

$$X(x_1, x_2, x_3) = (x_1' \wedge x_2 \wedge x_3) \vee (x_1' \wedge x_2 \wedge x_3') \vee (x_1 \wedge x_2 \wedge x_3) \vee (x_1 \wedge x_2' \wedge x_3)$$

which is a DNF.

Example 14: Obtain the sum of products in canonical form of the following expression

$(x_1 \vee x_2)' \vee (x_1' \wedge x_3)$ **[UPTU., MCA I Sem., 2001-02]**

Solution: $(x_1 \vee x_2)' = x_1' \wedge x_2'$ (De-morgain's law)

$= (x_1' \wedge x_2') \wedge I$ (identity law)

$= (x_1' \wedge x_2') \wedge (x_3 \vee x_3')$ (complementation law)

$= (x_1' \wedge x_2' \wedge x_3) \vee (x_1' \wedge x_2' \wedge x_3')$ (distributive law) …(1)

Again $(x_1' \wedge x_3) = (x_1' \wedge x_3) \wedge I$ (identity law)

$= (x_1' \wedge x_3) \wedge (x_2 \vee x_2')$ (complementation law)

$= (x_1' \wedge x_3 \wedge x_2) \vee (x_1' \wedge x_3 \wedge x_2')$ (distributive law)

$= (x_1' \wedge x_2 \wedge x_3) \vee (x_1' \wedge x_2' \wedge x_3)$ (commutative law) …(2)

Putting values from (1) and (2) the given expression,

$$= (x_1' \wedge x_2' \wedge x_3) \vee (x_1' \wedge x_2' \wedge x_3') \vee (x_1' \wedge x_2 \wedge x_3) \vee (x_1' \wedge x_2' \wedge x_3)$$

which is the requires D.N.F. of the given Boolean expression.

14. MAXTERM

A Boolean expression in k variables $x_1, x_2, \ldots x_k$ is called maxterm if it is of the form $y_1 \vee y_2 \vee \ldots \vee y_k$ i.e. a join or sum of exactly k distinct variables (i.e. literals none of which involves the same variables) where each y_j is a literal (either x_i or x_i') for $1 \le i \le k$ and $y_i \ne y_j$ for $i \ne j$.

15. CONJUNCTIVE NORMAL FORM (CNF)

A Boolean expression in k variables is in CNF if it is a meet or product of distinct maxterms, each involving all the k variables.

Illustration: The Boolean expression,

$X(x_1, x_2, x_3) = (x_1' \vee x_2 \vee x_3) \wedge (x_1' \vee x_2' \vee x_3) \wedge (x_1' \vee x_2 \vee x_3')$ is in CNF as it is the meet or product of 3 distinct maxterms each involving 3 variables.

Example 15: Obtain CNF of Boolean expression

$$X(x_1, x_2, x_3) = (x_1 \wedge x_2)' \wedge (x_1' \wedge x_3)' \quad \ldots(1)$$

Solution: $(x_1 \wedge x_2)' = x_1' \vee x_2' = (x_1' \vee x_2') \vee 0$

$= (x_1' \vee x_2') \vee (x_3 \wedge x_3')$

$= (x_1' \vee x_2' \vee x_3) \wedge (x_1' \vee x_2' \vee x_3')$ …(2)

[Applying De-Morgan, Identity, Complementation & distributive laws respectively)

Similarly $(x_1' \wedge x_3)' = (x_1 \vee x_2' \vee x_3') \wedge (x_1 \vee x_2 \vee x_3')$ …(3)

Using (2) and (3) we get R.H.S. of (1) as

$= (x_1' \vee x_2' \vee x_3) \wedge (x_1' \vee x_2' \vee x_3') \wedge (x_1 \vee x_2' \vee x_3') \wedge (x_1 \vee x_2 \vee x_3')$ which is in CNF.

Complete Conjunctive normal Form: A CNF in n variables which contains all the possible 2^n factors is called Complete CNF in n variables.

For example Complete CNF in 2 variables x, y is $(x + y)(x' + y)(x + y')(x' + y')$ (containing $2^2 = 4$ terms) and complete CNF in 3 variables x, y, z is $(x + y + z)(x' + y + z)(x + y' + z)(x + y + z')(x' + y' + z)(x + y' + z')(x' + y + z')(x' + y' + z')$, (containing $2^3 = 8$ terms)

Note: A complete CNF is identically 0.

16. EQUIVALENT BOOLEAN EXPRESSIONS

Definition: Two Boolean expression are equivalent if and only if their respective canonical forms are identical.

17. REDUCTION OF BOOLEAN EXPRESSION TO SIMPLER FORMS

Simpler form means, that the expression has fewer connectives and all the literals involved are distinct.

Example 16: Reduce to simpler form, the Boolean expressions

(a) $X(x_1, x_2) = (x_1 \wedge x_2) \wedge (x_1 \wedge x_2')$

(b) $X(x_1, x_2, x_3) = (x_1 \wedge x_2) \vee (x_1 \wedge x_2' \wedge x_3) \wedge (x_1 \wedge x_3)$

Solution: (a) R.H.S. $= ((x_1 \wedge x_2) \wedge x_1) \wedge x_2'$ (Associative law)

$= (x_1 \wedge x_2) \wedge x_2'$ (Absorptions law)

$= x_1 \wedge (x_2 \wedge x_2')$ (Associative law)

$= x_1 \wedge 0$ (Complementation law)

$= 0$ (Identity law)

(b) R.H.S.$= [x_1 \wedge \{x_2 \vee (x_2' \wedge x_3)\}] \wedge (x_1 \wedge x_3)$ (Distributive law)

$= [x_1 \wedge \{(x_2 \vee x_2') \wedge (x_2 \vee x_3)\}] \wedge (x_1 \wedge x_3)$ (Distributive law)

$= [x_1 \wedge \{I \wedge (x_2 \vee x_3)\}] \wedge (x_1 \wedge x_3)$ (Complementation law)

$= [x_1 \wedge (x_2 \vee x_3)] \wedge (x_1 \wedge x_3)$ (Identity law)

$= [(x_1 \wedge x_2) \vee (x_1 \wedge x_3)] \wedge (x_1 \wedge x_3)$ (Distributive law)

$= [(x_1 \wedge x_2) \wedge (x_1 \wedge x_3)] \vee (x_1 \wedge x_3) \wedge (x_1 \wedge x_3)$

$= (x_1 \wedge x_2 \wedge x_3) \vee (x_1 \wedge x_3)$ (Idempotent and Associative law)

$= (x_1 \wedge [(x_2 \wedge x_3) \vee x_3)]$ (Distributive law)

$= x_1 \wedge x_3$ (Absorption law)

Example 17: Simplify the following Boolean expressions:

(i) $x_1x_2 + x_1x_2'$ (ii) $xy + x'y' + x'y$ (iii) $x'y'z' + x'yz' + x'y'z + x'yz$

Solution: (i) $x_1x_2 + x_1x_2'$

$= x_1(x_2 + x_2') = x_1 \bullet 1 = x_1$

(ii) $xy + x'y' + x'y$

$= xy + x'y' + x'y + x'y$ (Idempotent law)

$= xy + x'(y' + y) + x'y$ (distributive law)

$= xy + x' \bullet 1 + x'y$ (complement law)

$= (x + x')\, y + x'$

$= 1 \bullet y + x'$

$= y + x'$

(iii) $x'y'z' + x'yz' + x'y'z + x'yz$

$= x'y'(z' + z) + x'y\,(z' + z)$ (distributive law)

$= x' \bullet y' \bullet 1 + x'\, y \bullet 1$ (complement law)

$= x'\, y' + x'\, y$ (identity law)

$= x'\,(y' + y)$

$= x' \bullet 1 = x'$

Note: Such type of simplification is also possible with the help of K-maps.

Example 18: Simplify the following Boolean expressions:

(i) $(a + b) \bullet a' \bullet b'$ (ii) $(a + a' \bullet b) \bullet (a' + a \bullet b)$ (iii) $a \bullet b + a' \bullet b' + a' \bullet b + a \bullet b'$

Solution: (i) $(a + b) \bullet a' \bullet b'$

$= a \bullet a' \bullet b' + b \bullet a' \bullet b'$ (by distributive law)

$= (a \bullet a') \bullet b' + (b \bullet b') \bullet a'$ (by associative and commutative laws)

$= 0 \bullet b' + 0 \bullet a'$ (as $a \bullet a' = 0$ and $b \bullet b' = 0$)

$= b' \bullet 0 + a' \bullet 0$

$= 0 + 0$

$= 0$

(ii) $(a + a' \bullet b) \bullet (a' + a \bullet b)$

$= (a + a') \bullet (a + b) \bullet (a' + a) \bullet (a' + b)$ (by distributive law)

$= 1 \bullet (a + b) \bullet 1 (a' + b)$

$= (a + b) \bullet (a' + b)$

$= (b + a) \bullet (b + a')$ (by commutative law)

$= b + a \bullet a'$ (by distributive law)

$= b + 0$

$= b$

(iii) $a \bullet b + a' \bullet b' + a'b + a \bullet b'$

$= a \bullet b + a \bullet b' + a' \bullet b + a' \bullet b'$ (by associative law)

$= a \bullet (b + b') + a' \bullet (b + b')$ (by distributive law)

$= a \bullet 1 + a' \bullet 1$

$= a + a'$

$= 1$ (as $a + a' = 1$)

Example 19: Simplify the following Boolean expressions:

(i) $x \bullet y + (x + y) \bullet z' + y$ (ii) $x + y + (x' + y + z)'$

Solution: (i) $x \bullet y + (x + y) \bullet z' + y$

$= (x \bullet y + y) + x \bullet z' + y \bullet z'$ (by distributive and associative law)

$= y + x \bullet z' + y \bullet z'$ (by absorption law)

$= (y + y \bullet z') + x \bullet z'$ (by associative law)

$= y + x \bullet z'$ (by absorption law)

(ii) $x + y + (x' + y + z)'$

$= (x + y) + x \bullet y' \bullet z'$ (by De Morgan's law)

$= (x + y + x) \bullet (x + y + y') \bullet (x + y \bullet z')$ (by distributive law)

$= (x + y) \bullet (x + 1) \bullet (x + y + z')$ (by idempotent and complement law)

$= (x + y) \bullet 1 \bullet (x + y + z')$ (by boundedness law)

$= (x + y) \bullet (x + y + z')$ (by identity law)

$= x + y$ (by absorption law)

Example 20: Simplify the following Boolean expression

$y \bullet z + w \bullet x + z + [w \bullet z \bullet (x \bullet y + w \bullet z)]$

Solution: $y \bullet z + w \bullet x + z + [w \bullet z \bullet (x \bullet y + w \bullet z)]$

$= (y \bullet z + z) + w \bullet x + w \bullet z \bullet x \bullet y + w \bullet z \bullet w \bullet z$

$= z + (w \bullet x + w \bullet x \bullet z \bullet y) + (w \bullet w) \bullet (z \bullet z)$ (by absorption law and commutative law)

$= z + w \bullet x + w \bullet z$

$= (z + z \bullet w) + w \bullet x$

$= z + w \bullet x$ (by absorption law)

Example 21: Simplify the following Boolean expressions:

(i) $(a \bullet b' + c)'$ (ii) $a \bullet b \bullet c + a' + b' + c'$

(iii) $a \bullet b + [(a + b') \bullet b]'$ (iv) $[(a' \bullet b')' + a] \bullet (a + b')'$

Solution: (i) $(a \bullet b' + c)'$

$= (a \bullet b')' \bullet c'$ (by De Morgan's law)

$= (a' + b) \bullet c'$ (by De Morgan's law)

(ii) $a \bullet b \bullet c + a' + b' + c'$

$= a \bullet b \bullet c + (a \bullet b \bullet c)'$ (by De Morgan's law)

$= 1$ (as $a + a' = 1$)

(iii) $a \bullet b + [(a + b') \bullet b]'$

$= a \bullet b + (a + b')' + b'$ (by De Morgan's law)

$= a \bullet b + a' \bullet b + b'$ (by De Morgan's law)

$= a \bullet b + (b' + a' \bullet b)$ (by Commutative law)

$= a \bullet b + (b' + a') \bullet (b' + b)$ (by distributive law)

$= a \bullet b + (b' + a') \bullet 1$

$= a \bullet b + a' + b'$

$= a \bullet b + (a \bullet b)'$

$= 1$ (as $a + a' = 1$)

(iv) $[(a' \bullet b') + a] \bullet (a + b')'$

$= [(a + b) + a] \bullet (a' \bullet b)$ (by De Morgan's law)

$= (a + a + b) \bullet (a' \bullet b)$ (by commutative law)

$= (a + b) \bullet (a' \bullet b)$ (as $a + a = a$)

$= a \bullet a' \bullet b + b \bullet a' \bullet b$ (by distributive law)

$= (a \bullet a') \bullet b + (b \bullet b) \bullet a'$ (by associative and commutative laws)

$= 0 \bullet b + b \bullet a'$ (as $a \bullet a' = 0$ and $b \bullet b = b$)

$= b \bullet 0 + a' b$

$= 0 + a' b$

$= a' b$

Example 22: Prove that :

$$(x_1' + x_2' + x_3' + x_4') \bullet (x_1' + x_2' + x_3' + x_4) \bullet (x_1' + x_2' + x_3 + x_4) \bullet (x_1' + x_2' + x_3 + x_4') = x_1' + x_2'$$

Solution: $(x_1' + x_2' + x_3' + x_4') \bullet (x_1' + x_2' + x_3' + x_4)$

$= x_1' + x_2' + x_3' + x_4' \bullet x_4$ (by distributive law)

$= x_1' + x_2' + x_3' + 0$

$= x_1' + x_2' + x_3'$ …(1)

Again $(x_1' + x_2' + x_3 + x_4) \bullet (x_1' + x_2' + x_3 + x_4')$

$= x_1' + x_2' + x_3 + x_4 \bullet x_4'$

$= x_1' + x_2' + x_3 + 0$

$= x_1' + x_2' + x_3$ …(2)

Combining (1) and (2) by operation.

$(x_1' + x_2' + x_3' + x_4') \bullet (x_1' + x_2' + x_3' + x_4) \bullet (x_1' + x_2' + x_3 + x_4) \bullet (x_1' + x_2' + x_3 + x_4')$

$= (x_1' + x_2' + x_3') \bullet (x_1' + x_2' + x_3)$

$= x_1' + x_2' + x_3' \bullet x_3$ (by distributive property)

$= x_1' + x_2' + 0$ (as $x \bullet x' = 0$)

$= x_1' + x_2'$ (by identity property)

18. CONVERSION OF GIVEN BOOLEAN EXPRESSIONS INTO EQUIVALENT DISJUNCTIVE NORMAL FORM OR SUM OF PRODUCT CANONICAL FORM

Example 23: Determine disjunctive normal form or sum-of-product canonical form equivalent to the following Boolean expressions in three variables:

(i) $(x + y'z) \bullet (y + z')$ (ii) $(x' + y)' + y' z$

(iii) $y (x + y z)'$ (iv) $x (x y' + x' y + y' z)$

(v) $(x' \bullet y)' \bullet (x' + x \bullet y \bullet z')$

Solution: (i) $(x + y' \bullet z) \bullet (y + z')$

$= x \bullet y + x \bullet z' + y' \bullet z \bullet y + y' \bullet z z'$

$= x y + x \bullet z' + z \bullet y \bullet y' + y' \bullet z \bullet z'$

$= x \bullet y + x \bullet z' + z \bullet 0 + y' \bullet 0$ (as $y \bullet y' = z \bullet z' = 0$)

$= x \bullet y + x \bullet z' + 0 + 0$ (by boundedness law)

$= x \bullet y + x \bullet z'$

$= x \bullet y \bullet 1 + x \bullet z' \bullet 1$

$= x \bullet y \bullet (z + z') + x \bullet z' (y + y')$

$= x \bullet y \bullet z + x y \bullet z' + x \bullet y \bullet z' + x \bullet y' z'$

$= x \bullet y \bullet z + x \bullet y \bullet z' + x \bullet y' z'$ (by idempotent law)

(ii) $(x' + y)' + y' \bullet z$

$= x \bullet y' + y' \bullet z$

$= x \bullet y' \bullet 1 + y' \bullet z \bullet 1$

$= x \bullet y' \bullet (z + z') + y' \bullet z \bullet (x + x')$

$= x \bullet y' z + x y' z' + x \bullet y' \bullet z + x' y' z$

$= x \bullet y' \bullet z + x y' z' + x' \bullet y' \bullet z$ (by idempotent law)

(iii) $y(x + y \bullet z)'$

$= y \bullet \left(x' \bullet (y \bullet z)'\right)$ (by De Morgan's law)

$= y \bullet x' \bullet (y' + z')$ (by De Morgan's law)

$= x' \bullet y \bullet y' + x' \bullet y \bullet z'$ (by commutative and distributive law)

$= x' \bullet 0 + x' \bullet y \bullet z'$

$= 0 + x' \bullet y \bullet z'$ (by boundedness law)

$= x' \bullet y \bullet z'$

(iv) $x(x \bullet y' + x' y + y' \bullet z)$

$= x \bullet x \bullet y' + x \bullet x' \bullet y + x \bullet y' \bullet z$

$= x \bullet y' + 0 \bullet y + x \bullet y' \bullet z$ (by idempotent and identity laws)

$= x \bullet y' \bullet 1 + y \bullet 0 + x \bullet y' \bullet z$

$= x \bullet y' \bullet (z + z)' + 0 + x \bullet y' \bullet z$ (by identity and boundedness laws)

$= x \bullet y' \bullet z + x \bullet y' \bullet z' + x \bullet y' \bullet z$

$= x \bullet y' \bullet z + x \bullet y' \bullet z + x \bullet y' \bullet z'$ (by associative law)

$= x \bullet y' \bullet z + x \bullet y' \bullet z'$ (by idempotent law)

(v) $(x' \bullet y)' \bullet (x' + x \bullet y \bullet z')$

$= (x + y') \bullet (x' + x \bullet y \bullet z')$ (by De Morgan's law)

$= x \bullet x' + x \bullet x \bullet y \bullet z' + x' \bullet y' + x \bullet yy' \bullet z'$ (by distributive and commutative laws)

$= 0 + x \bullet y \bullet z' + x' \bullet y' \bullet 1 + x \bullet 0 \bullet z'$ (by identity and idempotent law)

$= x \bullet y \bullet z' + x' \bullet y' \bullet (z + z)' + 0$ (by identity and boundedness law)

$= x \bullet y \bullet z' + x' \bullet y' \bullet z + x' \bullet y' z'$

Example 24: Express the following functions in their equivalent disjunctive normal forms:

(i) $x + x' \bullet y$ (ii) $(x y' + x z)' + x'$

Solution: (i) $x + x' \bullet y$

$= x \bullet 1 + x' \bullet y$

$= x \bullet (y + y') + x' \bullet y$

$= x \bullet y + x \bullet y' + x' \bullet y$

(ii) $(x \bullet y' + x \bullet z)' + x'$

$= (x \bullet y')' \bullet (x \bullet z)' + x'$ (by De Morgan's law)

$= (x' + y) \bullet (x' + z') + x'$ (by De Morgan's law)

$= x' \bullet x' + y \bullet x' + x' \bullet z' + y \bullet z' + x'$ (by distributive law)

$= (x' + x' \bullet y) + yz' + (x' z' + x')$ (by commutative and associative laws)

$= x' + y \bullet z' + x'$ (by absorption law)

$= x' + x' + y \bullet z'$ (by commutative law)

$= x' + y \bullet z'$ $(x' + x' = x)$

$= x' \bullet 1 \bullet 1 + y \bullet z' \bullet 1$

$= x' \bullet (y + y') \bullet (z + z') + y \bullet z' \bullet (x + x')$ (as $x + x' = y + y' = z + z' = 1$)

$= x' \bullet (y \bullet z + y' z + y \bullet z' + y' \bullet z') + y \bullet z' \bullet x + y \bullet z' \bullet x'$

$= x' \bullet y \bullet z + x' \bullet y' \bullet z + x' \bullet y \bullet z' + x' \bullet y' \bullet z' + x \bullet y \bullet z' + x' \bullet y \bullet z'$

$= x' \bullet y \bullet z + x' \bullet y' \bullet z + x' \bullet y \bullet z' + x' \bullet y' \bullet z' + x \bullet y \bullet z'$ $\left[\begin{array}{l}\text{as } x' \bullet y \bullet z' + x' \bullet y \bullet z' \\ = x' \bullet y\, z'\end{array}\right]$

which is in D.N.F.

Example 25: Express the following Boolean expressions into their equivalent disjunctive normal form in x, y and z

(i) $x + y$ (ii) x (iii) $(z + y) \bullet (z' + y')$

(iv) $x \bullet y \bullet z + x + x\, z + y \bullet z$

Solution: (i) $x + y$

$= x \bullet 1 \bullet 1 + y \bullet 1 \bullet 1$

$= x \bullet (y + y') \bullet (z + z') + y \bullet (x + x') \bullet (z + z')$

$= x \bullet (y \bullet z + y \bullet z' + y' \bullet z + y' z') + y \bullet (x \bullet z + x \bullet z' + x' \bullet z + x' \bullet z')$ (by distributive law)

$= x \bullet y \bullet z + x \bullet y \bullet z' + x \bullet y' \bullet z + x \bullet y' \bullet z'$

$+ x \bullet y \bullet z + x \bullet y \bullet z' + x' \bullet y \bullet z + x' \bullet y \bullet z'$ (by distributive and commutative laws)

$= x \bullet y \bullet z + x \bullet y \bullet z' + x \bullet y' \bullet z + x \bullet y' \bullet z' + x' \bullet y \bullet z + x' \bullet y \bullet z'$

which is in D.N.F.

(ii) $x = x \bullet 1 \bullet 1$

$= x \bullet (y + y') \bullet (z + z')$

$= x \bullet (y \bullet z + y \bullet z' + y' z + y' \bullet z')$ (by distributive law)

$= x \bullet y \bullet z + x \bullet y \bullet z' + x \bullet y' \bullet z + x \bullet y' \bullet z'$ (by distributive law)

which is in D.N.F.

(iii) $(z + y) \bullet (z' + y')$

$= z \bullet z' + z \bullet y' + y \bullet z' + y\, y'$

$= 0 + z \bullet y' + y \bullet z' + 0$

$= z \bullet y' \bullet 1 + y \bullet z' \bullet 1$

$= (z \bullet y') \bullet (x + x') + y \bullet z' \bullet (x + x')$

$= x \bullet y' \bullet z + x' \bullet y' z + x \bullet y \bullet z' + x' \bullet y \bullet z'$ (by distributive and commutative law)

which is in D.N.F.

(iv) $x \bullet y \bullet z + x + x \bullet z + y\, z$

$= x \bullet y \bullet z + x \bullet 1 \bullet 1 + x \bullet z \bullet 1 + y \bullet z \bullet 1$

$= x \bullet y \bullet z + x \bullet (y + y') \bullet (z + z') + x \bullet z\, (y + y') + y \bullet z \bullet (x + x')$

$= x \bullet y \bullet z + x \bullet y \bullet z + x \bullet y' \bullet z + x \bullet y \bullet z' + x\, y' \bullet z' + x\, z \bullet y + x \bullet z \bullet y'$

$+ y \bullet z\, x + y \bullet z \bullet x'$ (by distributive law)

$= x \bullet y \bullet z + x \bullet y' \bullet z + x \bullet y \bullet z' + x \bullet y' \bullet z' + x \bullet y' \bullet z + x' \bullet y \bullet z$ (by commutative law)

Also $(x \bullet y \bullet z + x \bullet y \bullet z = x \bullet y \bullet z)$

Example 26: Convert the following Boolean expressions into equivalent sum of products cononical form in three variables x, y and z:

(i) $x \bullet y$ (ii) $(x \bullet y)' + z$

Solution: (i) $x \bullet y = x \bullet y \bullet 1 = x \bullet y \bullet (z + z')$

$= x \bullet y\, z + x \bullet y \bullet z'$

which is in D.N.F.

(ii) $(x \bullet y)' + z$

$= x' + y' + z$

$= x' \bullet 1 \bullet 1 + y' \bullet 1 \bullet 1 + z \bullet 1 \bullet 1$

$= x' (y + y') (z + z') + y' (z + z') (x + x') + z (x + x') (y + y')$

$= x' (yz + yz' + y'z + y'z') + y' (x z + x'z + x z' + x'z') + z (x y + x y' + x' y + x' y')$

$= x' y z + x' y z' + x' y' z + x' y' z' + x y' z + x y z + x' y z + x' y' z + x' y' z' + x y' z'$
$+ x' y' z + x y' z$

$= x' y z + x' y z' + x' y' z + x' y'z' + x y' z' + x y' z + x y z$

which is in D.N.F.

Example 27: Find disjunctive normal form equivalent to

$(x + y + z) (x \bullet y + x' z')'$ **[C.C.S.U., M.Sc. (Maths) 2004]**

Solution: $(x + y + z) (x \bullet y + x' \bullet z')' = (x + y + z) \bullet [(x y)' \bullet (x' \bullet z')']$

$= (x + y + z) [(x' + y') \bullet (x + z)]$ (by De Morgan law)

$= (x + y + z) \bullet (x' \bullet x + x' \bullet z + y' x + y' z)$ (by distributive law)

$= (x + y + z) \bullet (x \bullet x' + x' \bullet z + x \bullet y' + y' z)$ (by commutative law)

$= (x + y + z) (0 + x' \bullet z + x \bullet y' + y' \bullet z)$ (as $x \bullet x' = 0$)

$= (x + y + z) (x' \bullet z + x \bullet y' + y' \bullet z)$ (identity property)

$= x \bullet x' z + x \bullet x \bullet y' + x \bullet y' \bullet z + x' y \bullet z + x y \bullet y' + y \bullet y' \bullet z$
$+ x' \bullet z \bullet z + x \bullet y' z + y' \bullet z \bullet z$ (by distributive and commutative law)

$= 0 \bullet z + x \bullet y' + x \bullet y' \bullet z + x' \bullet y \bullet z + x \bullet 0 + 0 \bullet z + x' \bullet z + x \bullet y' z + y' \bullet z$
(as $x \bullet x' = y \bullet y' = z \bullet z' = 0$ and $x \bullet x = x$, $y \bullet y = y$ and $z \bullet z = z$)

$= x \bullet y' + x \bullet y' z + x' \bullet y \bullet z + x' \bullet z + x \bullet y' \bullet z + y' \bullet z$ as $0 \bullet z = 0 \bullet x = 0$

$= x \bullet y' \bullet 1 + x \bullet y' \bullet z + x' \bullet y \bullet z + x' \bullet z \bullet 1 + y' \bullet z \bullet 1$

$= x \bullet y' (z + z') + x \bullet y' \bullet z + x' \bullet y \bullet z + x' \bullet z (y + y') + y' \bullet z (x + x')$

$= x \bullet y' \bullet z + x \bullet y' z' + x \bullet y' \bullet z + x' \bullet y \bullet z + x' \bullet y \bullet z + x' y' z + x \bullet y' z + x' y' z)$

$= x \bullet y' \bullet z + x \bullet y' \bullet z' + x' \bullet y \bullet z + x' \bullet y' \bullet z$ which is in DNF.

Example 28: Express the following expressions in terms of their equivalent disjunctive normal form

(i) $(u + v)(u + v')(u' + w)$ (ii) $(u + v')(v + w')(w + u')(u' + v')$

Solution: (i) $(u + v) \bullet (u + v') \bullet (u' + w)$

$= (u + v \bullet v') \bullet (u' + w)$ (by distributive law)

$= (u + 0) \bullet (u' + w)$ (by inverse property)

$= u \bullet (u' + w)$ (by identity property)

$= u \bullet u' + u \bullet w$ (by distributive property)

$= 0 + u \bullet w$

$= u \bullet w$

$= u \bullet w \bullet 1$

$= u \bullet w \bullet (v + v')$

$= u \bullet w \bullet v \bullet + u \bullet w \bullet v'$

$= u \bullet v \bullet w + u \bullet v' \bullet w$

(ii) $(u + v') \bullet (v + w') \bullet (w + u') \bullet (u' + v')$

$= (u \bullet v + v' \bullet v + u \bullet w' + v' \bullet w') \bullet (u' + w) \bullet (u' + v')$ (by distributive and commutive laws)

$= (u \bullet v + 0 + u \bullet w' + v' \bullet w') \bullet (u' + w \bullet v')$ (by inverse law and distributive law)

$= (u \bullet v + u \bullet w' + v' \bullet w') \bullet (u' + w \bullet v')$

$= u \bullet v \bullet u' + u \bullet w' \bullet u' + v' \bullet w' \bullet u' + u \bullet v \bullet w \bullet v' + u \bullet w' \bullet w \bullet v' + v' \bullet w' \bullet w \bullet v'$

$= u \bullet u' \bullet v + u \bullet u' \bullet w' + u' \bullet v' \bullet w' + u \bullet v \bullet v' \bullet w + u \bullet v' \bullet w \bullet w' + v' \bullet v' \bullet w \bullet w'$

(by commutative law)

$= 0 \bullet v + 0 \bullet w' + u' \bullet v' \bullet w' + u \bullet o \bullet w + u \bullet v' \bullet 0 + v' \bullet 0$

(by inverse property and $v' \bullet v' = v'$)

$= 0 + 0 + u' \bullet v' \bullet w' + 0 + 0 + 0$ (by commutative and boundedness law)

$= u' \bullet v' \bullet w'$ (by identity property)

19. VALUE OF A MINTERM

A minterm has the value 1 for one and only one combination of values of the variables.

The value of minterm $y_1 \bullet y_2 \bullet \ldots \bullet y_n$ is 1 which occurs.

if and only if each y_i is 1

or if and only if $x_i = 1$ when $y_i = x_i$ and $x_i = 0$ when $y_i = x_i'$

Example 29: Find a minterm which is equal to 1 when $x_1 = x_4 = 0$ and $x_2 = x_3 = x_5 = 1$ and which is equal to 0 otherwise.

Solution: The required minterm shall be $x_1' \bullet x_2 \bullet x_3 \bullet x_4' \bullet x_5$

Example 30: Convert the Boolean junction $f(x, y, z) = (x + y) \bullet z'$ into sum of products expansion or the disjunctive normal form by using truth table.

Solution: The value of the function f(x, y, z) are determined in the table given below:

x	y	z	x + y	z′	(x + y) • z′
0	0	0	0	1	0
0	0	1	0	0	0
0	1	0	1	1	1
0	1	1	1	0	0
1	0	0	1	1	1
1	0	1	1	0	0
1	1	0	1	1	1
1	1	1	1	0	0

The value of the function is 1 in 3^{rd}, 5^{th} and 7^{th} rows in which values of x, y and z are (0, 1, 0), (1, 0, 0) and (1, 1, 0) which corresponds to the minterm $x' \bullet y \bullet z'$, $x \bullet y' \bullet z'$ and $x \bullet y \bullet z'$.
Therefore the required function can be written as sum of these minterm as given below:

$$(x + y) \bullet z' = x' \bullet y \bullet z' + x \bullet y' \bullet z' + x \bullet y \bullet z'$$

which is in D.N.F.

20. CONVERSION OF GIVEN BOOLEAN EXPRESSIONS INTO EQUIVALENT CONJUCTIVE NORMAL OR PRODUCT OF SUM CANONICAL FORM

Example 31: Determine conjunctive normal form or product of sum canonical form equivalent to the following Boolean expressions in three variables x, y and z.

(i) $x + y$ (ii) $x \bullet y$ (iii) $y \bullet (z' \bullet x)'$

(iv) $u \bullet v \bullet w + (u + v) \bullet (u + w)$ (v) $(u \bullet v' + u \bullet w)' + u'$

Solution: (i) $x + y$

$= x + y + 0$ (by identity law)

$= (x + y) + z \bullet z'$ (by complementation law)

$= (x + y + z) \bullet (x + y + z')$ (by distributive law)

(ii) $x \bullet y$

$= (x + 0) \bullet (y + 0)$ (by identity law)

$= (x + y \bullet y') \bullet (y + x \bullet x')$ (by complementation law)

$= (x + y) \bullet (x + y') \bullet (y + x) \bullet (y + x')$ (by distributive law)

$= (x + y) \bullet (x + y) \bullet (x + y') \bullet (x' + y)$ (by associative and commutative law)

$= (x + y) \bullet (x + y') \bullet (x' + y)$ (by idempotent law)

$= (x + y + 0) \bullet (x + y' + 0) \bullet (x' + y + 0)$

$= (x + y + z \bullet z') \bullet (x + y' + z \bullet z') \bullet (x' + y + z \bullet z')$

$= (x + y + z) \bullet (x + y + z') \bullet (x + y' + z) \bullet (x + y' + z') \bullet (x' + y + z) \bullet (x' + y + z')$

(iii) $y \bullet (z' \bullet x)'$

$= y \bullet (z + x')$ (by distributive law)

$= (y + 0) \bullet [(z + x') + 0]$

$= (y + z \bullet z') \bullet [(z + x') + y \bullet y']$
$= (y + z) \bullet (y + z') \bullet (z + x' + y) \bullet (z + x' + y')$
$= (y + z + 0) \bullet (y + z' + 0) \bullet (z + x' + y) \bullet (z + x' + y')$
$= (y + z + x \bullet x') \bullet (y + z' + x \bullet x') \bullet (x' + y + z) \bullet (x' + y' + z)$
$= (y + z + x) (y + z + x') \bullet (y + z' + x) \bullet (y + z' + x') \bullet (x' + y + z) (x' + y' + z)$
$= (x + y + z) \bullet (x' + y + z) \bullet (x + y + z') \bullet (x' + y + z') \bullet (x' + y + z) \bullet (x' + y' + z)$
$= (x + y + z) \bullet (x' + y + z) \bullet (x + y + z') \bullet (x' + y + z') \bullet (x' + y' + z)$

(iv) $u \bullet v \bullet w + (u + v) \bullet (u + w)$
$= u \bullet v \bullet w + u + v \bullet w$ (by distributive law)
$= (u + u \bullet v \bullet w) + v \bullet w$ (by commutative law)
$= u + v \bullet w$ (by absorption law)
$= (u + v) \bullet (u + w)$ (by distributive law)
$= (u + v + 0) \bullet (u + w + 0)$ (identity property)
$= (u + v + w \bullet w') \bullet (u + w + v \bullet v')$ (inverse property)
$= (u + v + w) \bullet (u + v + w') \bullet (u + w + v) \bullet (u + w + v')$ (by distributve law)
$= (u + v + w) \bullet (u + v + w') \bullet (u + v' + w)$

which is in C.N.F.

(v) $(u \bullet v' + u \bullet w)' + u'$
$= (u \bullet v')' \bullet (u \bullet w)' + u'$ (by De Morgan's law)
$= (u' + v) \bullet (u' + w') + u'$ (by De Morgan's law)
$= (u' + v + u') \bullet (u' + w' + u')$ (by distributive law)
$= (u' + u' + v) \bullet (u' + u' + w')$ (by commutative law)
$= (u' + v) \bullet (u' + w')$
$= (u' + v + 0) \bullet (u' + w' + 0)$ (identity property)
$= (u' + v + w \bullet w') \bullet (u' + w' + v \bullet v')$ (inverse property)
$= (u' + v + w) \bullet (u' + v + w') \bullet (u' + w' + v) \bullet (u' + w' + v')$ (by distributive property)
$= (u' + v + w) \bullet (u' + v + w') \bullet (u' + w' + v')$

Which in C.N.F.

Example 32: Determine conjunctive normal form or product of sum canonical form equivalent to the following Boolean expressions in three variables:

(i) x (ii) $x + y'$ (iii) $y' z + y z'$
(iv) $u + u' \bullet v$ (v) $u \bullet v + u \bullet v' + u \bullet w$

Solution: (i) x
$= x + 0$ (identity property)
$= x + y \bullet y'$ (inverse property)
$= (x + y) \bullet (x + y')$ (by distributive law)
$= (x + y + 0) \bullet (x + y' + 0)$ (identity property)
$= (x + y + z \bullet z') \bullet (x + y' + z \bullet z')$ (inverse property)
$= (x + y + z) \bullet (x + y + z') \bullet (x + y' + z) \bullet (x + y' + z')$ (by distributive law)

which is in C.N.F.

(ii) $x + y'$

$= x + y' + 0$ (identity property)

$= x + y' + z \cdot z'$ (inverse property)

$= (x + y' + z) \cdot (x + y' + z')$ (by distributive law)

(iii) $y' \cdot z + y \cdot z'$

$= y' \cdot z + 0 + y \cdot z' + 0$ (by identity property)

$= y' \cdot z + y \cdot y' + y \cdot z' + z \cdot z'$ (by inverse property)

$= (z \cdot y' + y \cdot y') + (y \cdot z' + z \cdot z')$ (by commutative law)

$= (z + y) \cdot y' + (y + z) \cdot z'$ (by distributve law)

$= (z + y) \cdot (y' + z')$ (by distributive law)

$= (z + y + 0) \cdot (y' + z' + 0)$ (identity property)

$= (z + y + x \cdot x') \cdot (y' + z' + x \cdot x')$ (inverse property)

$= (z + y + x) \cdot (z + y + x') \cdot (y' + z' + x) \cdot (y' + z' + x')$ (by distributive property)

$= (x + y + z) \cdot (x' + y + z) \cdot (x + y' + z') \cdot (x' + y' + z')$ (by commutative law)

which is in C.N.F.

(iv) $u + u' \cdot v$

$= (u + u') \cdot (u + v)$ (by distributive law)

$= 1 \cdot (u + v)$ (inverse property)

$= u + v$ (identity property)

$= (u + v + 0)$ (identity property)

$= (u + v + w \cdot w')$ (inverse property)

$= (u + v + w) \cdot (u + v + w')$ (by distributive law)

(v) $u \cdot v + u \cdot v' + u \cdot w$

$= u \cdot (v + v') + u \cdot w$ (by distributive law)

$= u \cdot 1 + u \cdot w$ (inverse property)

$= u + u \cdot w$ (identity property)

$= u$ (by absorption law)

$= (u + 0)$ (identity property)

$= (u + v \cdot v')$ (inverse property)

$= (u + v) \cdot (u + v')$ (by distributive law)

$= (u + v + 0) \cdot (u + v' + 0)$ (identity property)

$= (u + v + w \cdot w') \cdot (u + v' + w \cdot w')$ (inverse property)

$= (u + v + w) \cdot (u + v + w') \cdot (u + v' + w) \cdot (u + v' + w')$ (by distributive law)

which is in C.N.F.

Example 33: Determine conjunctive normal form equivalent to the following given function

$$f(a, b, c, d) = (a' + b' \cdot c + b \cdot c \cdot d' + b \cdot c' \cdot d + b' \cdot c' \cdot d' \cdot a)'$$

Solution: The given function is

$(a' + b' \cdot c + b \cdot c \cdot d' + b \cdot c' \cdot d + b' \cdot c' \cdot d' \cdot a)'$

$= a \cdot (b + c') \cdot (b' + c' + d) \cdot (b' + c + d') \cdot (b + c + d + a')$ …(1)

(by applying De Morgan's law twice)

$$= (a + 0) \bullet (b + c' + 0) \bullet (b' + c' + d + 0) \bullet (b' + c + d' + 0) \bullet (b + c + d + a')$$

(identity property)

$$= (a + b \bullet b') \bullet (b + c' + d \bullet d') \bullet (b' + c' + d + a \bullet a') \bullet (b' + c + d' + a \bullet a') \bullet (b + c + d + a')$$

(by inverse property, $0 = b \bullet b' = d \bullet d' = a \bullet a'$)

$$= (a + b) \bullet (a + b') \bullet (b + c' + d) \bullet (b + c' + d') \bullet (b' + c' + d + a) \bullet (b' + c' + d + a')$$
$$\bullet (b' + c + d' + a) \bullet (b' + c + d' + a') \bullet (b + c + d + a')$$

(applying distributive law to the first four factors)

$$= (a + b + 0) \bullet (a + b' + 0) \bullet (b + c' + d + 0) \bullet (b + c' + d' + 0) \bullet (b' + c' + d + a)$$
$$\bullet (b' + c' + d + a') \bullet (b' + c + d' + a) \bullet (b + c + d' + a') \bullet (b + c + d + a')$$

(using identity property in the first four factors)

$$= (a + b + c \bullet c') \bullet (a + b' + c \bullet c') \bullet (b + c' + d + a \bullet a') \bullet (b + c' + d' + a \bullet a') \bullet (b' + c' + d + a)$$
$$\bullet (b' + c' + d + a') \bullet (b' + c + d' + a) \bullet (b + c + d' + a') \bullet (b + c' + d + a')$$

(using inverse property in the first four factors)

$$= (a + b + c) \bullet (a + b + c') \bullet (a + b' + c) \bullet (a + b' + c') \bullet (b + c' + d + a) \bullet (b + c' + d + a')$$
$$\bullet (b + c' + d' + a) \bullet (b + c' + d' + a') \bullet (b' + c' + d + a) \bullet (b' + c' + d + a')$$
$$\bullet (b' + c + d' + a) \bullet (b' + c + d' + a') (b + c + d + a')$$

(using distributive law in first four factors)

$$= (a + b + c + 0) \bullet (a + b + c' + 0) \bullet (a + b' + c + 0) \bullet (a + b' + c' + 0)$$
$$\bullet (a + b + c' + d) \bullet (a' + b + c' + d) \bullet (a + b + c' + d') \bullet (a' + b + c' + d') \bullet (a + b' + c' + d)$$
$$\bullet (a' + b' + c' + d') \bullet (a + b' + c + d') \bullet (a' + b' + c + d') \bullet (a' + b + c + d)$$

(using identity property in the first four factors and applying commutative law on the remaining factors)

$$= (a + b + c + d \bullet d') \bullet (a + b + c' + d \bullet d') \bullet (a + b' + c + d \bullet d') \bullet (a + b' + c' + d \bullet d')$$
$$\bullet (a + b + c' + d) \bullet (a' + b + c' + d) \bullet (a + b + c' + d') \bullet (a' + b + c' + d')$$
$$\bullet (a + b' + c' + d) \bullet (a' + b' + c' + d) \bullet (a + b' + c + d') \bullet (a' + b' + c + d')$$
$$\bullet (a' + b + c + d)$$

(using inverse property in the first four factors, $0 = d \bullet d'$)

$$= (a + b + c + d) \bullet (a + b + c + d') \bullet (a + b + c' + d) \bullet (a + b + c' + d')$$
$$\bullet (a + b' + c + d) \bullet (a + b' + c + d') \bullet (a + b' + c' + d) \bullet (a + b' + c' + d')$$
$$\bullet (a + b + c' + d) \bullet (a' + b + c' + d) \bullet (a + b + c' + d') \bullet (a' + b + c' + d')$$
$$\bullet (a + b' + c' + d) \bullet (a' + b' + c' + d) \bullet (a + b' + c + d') \bullet (a' + b' + c + d')$$
$$\bullet (a' + b + c + d)$$

(using distributive law in the first four factors)

$$= (a + b + c + d) \bullet (a + b + c + d') \bullet (a + b + c' + d) \bullet (a + b + c' + d')$$
$$\bullet (a + b' + c + d) \bullet (a + b' + c + d') \bullet (a + b' + c' + d) \bullet (a + b' + c' + d')$$
$$\bullet (a' + b + c' + d) \bullet (a' + b + c' + d') \bullet (a' + b' + c' + d) \bullet (a + b' + c + d')$$
$$\bullet (a' + b + c + d)$$

(combining 9^{th} factor with 3^{rd} factor, 11^{th} factor with 4^{th} factor, 13^{th} factor with 7th factor, 15^{th} factor with 6^{th} factor by idempotent law, $a \bullet a = a$)

Alternatively we can also simplify the given expression (1) as follows:

First factors

$$= a = (a + 0) + (a + b \cdot b') = (a + b) \cdot (a + b')$$
$$= (a + b + 0) \cdot (a + b' + 0) = (a + b + c \cdot c') \cdot (a + b' + c \cdot c')$$
$$= (a + b + c) \cdot (a + b + c') \cdot (a + b' + c) \cdot (a + b' + c')$$
$$= (a + b + c + 0) \cdot (a + b + c' + 0) \cdot (a + b' + c + 0) \cdot (a + b' + c' + 0)$$
$$= (a + b + c + d \cdot d') \cdot (a + b + c' + d \cdot d') \cdot (a + b' + c + d \cdot d') \cdot (a + b' + c' + d \cdot d')$$
$$= (a + b + c + d) \cdot (a + b + c + d') \cdot (a + b + c' + d) \cdot (a + b + c' + d')$$
$$\cdot (a + b' + c + d) \cdot (a + b' + c + d') \cdot (a + b' + c' + d) \cdot (a + b' + c' + d')$$

Second factor

$$= (b + c') = (b + c' + 0) = (b + c' + d \cdot d')$$
$$= (b + c' + d) \cdot (b + c' + d') = (b + c' + d + 0) \cdot (b + c' + d' + 0)$$
$$= (b + c' + d + a \cdot a') \cdot (b + c' + d' + a \cdot a')$$
$$= (b + c' + d + a) \cdot (b + c' + d + a') \cdot (b + c' + d' + a) \cdot (b + c' + d' + a')$$
$$= (a + b + c' + d) \cdot (a' + b + c' + d) \cdot (a + b + c' + d') \cdot (a' + b + c' + d')$$

Third factor

$$= (b' + c' + d) = (b' + c' + d + 0) = (b' + c' + d + a \cdot a')$$
$$= (b' + c' + d + a) \cdot (b' + c' + d + a')$$
$$= (a + b' + c' + d) \cdot (a' + b' + c' + d)$$

Fourth factor

$$= (b' + c + d') = (b' + c + d' + 0) = (b' + c + d' + a \cdot a')$$
$$= (b' + c + d' + a) \cdot (b' + c + d' + a')$$
$$= (a + b' + c + d') \cdot (a' + b' + c + d')$$

Fifth factor

$$= (b + c + d + a') = (a' + b + c + d)$$

Combining these five factors and using idempotent law we get the answer.

Example 34: Find out the conjuctive normal form of the polynomial.

(i) $F(x, y, z) = (x + y + z) \cdot (x \cdot y + x' \cdot z)'$ **[C.C.S.U., M.Sc. (Maths) 2004]**

(ii) $(x y' + x z)' + x'$

Solution: (i) $(x + y + z) \cdot (x \cdot y + x' \cdot z)'$

$$= (x + y + z) \cdot [(x \cdot y)' \cdot (x' \cdot z)'] \quad \text{(by De Morgan's law)}$$
$$= (x + y + z) \cdot [(x' + y') \cdot (x + z')] \quad \text{(by De Morgan's law)}$$
$$= (x + y + z) \cdot (x' + y' + 0) \cdot (x + z' + 0)$$
$$= (x + y + z) \cdot (x' + y' + z \cdot z') \cdot (x + z' + y \cdot y')$$
$$= (x + y + z) \cdot (x' + y' + z) \cdot (x' + y' + z') \cdot (x + z' + y) \cdot (x + z' + y') \quad \left(\text{by distributive law}\right)$$
$$= (x + y + z) \cdot (x' + y' + z) \cdot (x' + y' + z') \cdot (x + y + z') \cdot (x + y' + z')$$

which is the required C.N.F.

(ii) $(x \cdot y' + x \cdot z)' + x'$

$$= (x \cdot y')' \cdot (xz)' + x' \quad \text{(by De Morgan's law)}$$

$= (x' + y) \bullet (x' + z') + x'$ (by De Morgan's law)

$= (x' + y + x') \bullet (x' + z' + x')$ (by distribution law)

$= (x' + y) \bullet (x' + z')$ (by commutative law and idempotent law)

$= (x' + y + 0 \bullet (x' + z' + 0)$ (identity property)

$= (x' + y + z \bullet z') \bullet (x' + z' + y \bullet y')$ (inverse property)

$= (x' + y + z) \bullet (x' + y + z') \bullet (x' + z' + y) \bullet (x' + z' + y')$ (by distributive property)

$= (x' + y + z) \bullet (x' + y + z') \bullet (x' + y + z') \bullet (x' + y' + z')$ which is in the required form.

$= (x' + y + z) \bullet (x' + y + z') (x' + y' + z')$

20.1 Value of a Max term

A max term has the value 0 for one and only one combination of values of its variables.

The max term $y_1 + y_2 + \ldots + y_n$ is equal to 0 which occurs

if and only if each $y_i = 0$

or if and only if $x_i = 1$ when $y_i = x_1'$ and $x_i = 0$ when $y_i = x_i'$.

Example 35: Find a max term which is equal to 0 when

$x_1 = x_4 = 0$ and $x_2 = x_3 = x_5 = 1$

and which is equal to 1 otherwise.

Solution: The required max term is

$x_1 + x_2' + x_3' + x_4 + x_5'$

Example 36: Convert the Boolean functions f(x, y, z) into product of sums expansion or conjuctive normal form by finding value of the function:

(i) x (ii) $x \bullet y$ (iii) $(x + z) \bullet y$

Solution: (i) $f(x, y, z) = x$

The values of f (x, y, z) are given below:

x	y	z	f (x,y,z) = x
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

The value of the function f (x, y, z) is 0 in rows 1, 2, 3 and 4.

Values of x, y and z in row 1 are 0, 0 and 0 which corresponds to the max term x + y + z.

Values of x, y, z in row 2 are 0, 0 and 1 which corresponds to max term $x + y + z'$.

Value of x, y and z is row 3 are 0, 1 and 0 which corresponds to max term $x + y' + z$.

Values of x, y and z in row 4 are 0, 1 and 1 which corresponds to max term $x + y' + z'$.

The required Boolean expression in CNF shall be the product of all these max term.

Hence the required CNF is given by

$$f(x, y, z) = (x + y + z) \bullet (x + y + z') \bullet (x + y' + z) \bullet (x + y' + z')$$

(ii)

x	y	z	x • y
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1

The value of the function $f(x, y, z) = x \bullet y$ is 0 in rows 1, 2, 3, 4, 5 and 6 in which values of the variables x, y and z and the corresponding max term are as follows:

Row	Values of (x, y, z)	Corresponding max term
1	0, 0, 0	$x + y + z$
2	0, 0, 1	$x + y + z'$
3	0, 1, 0	$x + y' + z$
4	0, 1, 1	$x + y' + z'$
5	1, 0, 0	$x' + y + z$
6	1, 0, 1	$x' + y + z'$

The required CNF shall be the product of all these max term as given below

$$x \bullet y = (x + y + z) \bullet (x + y + z') \bullet (x + y' + z) \bullet (x + y' + z') \bullet (x' + y + z) \bullet (x' + y + z')$$

(iii) $f(x, y, z) = (x + z) \bullet y$

The table giving values of the function is shown below:

x	y	z	x + z	(x + z) • y
0	0	0	0	0
0	0	1	1	0
0	1	0	0	0
0	1	1	1	1
1	0	0	1	0
1	0	1	1	0
1	1	0	1	1
1	1	1	1	1

The value of the function is 0 in rows 1, 2, 3, 5 and 6 in which the values of the variables x, y and z and their corresponding max terms are given below:

Row	Values of (x, y z)	Corresponding max term
1	(0, 0, 0)	$x + y + z$
2	(0, 0, 1)	$x + y + z'$
3	(0, 1, 0)	$x + y' + z$
5	(1, 0, 0)	$x' + y + z$
6	(1, 0, 1)	$x' + y + z'$

The required CNF is the product of these max term as given by

$$(x + z) \bullet y = (x + y + z) \bullet (x + y + z') \bullet (x + y' + z) \bullet (x' + y + z) \bullet (x' + y + z')$$

21. INTER-CONVERSION OF D.N.F AND C.N.F.

Example 37: Convert the following disjunctive normal forms into their equivalent conjunctive normal forms in three variables:

(i) $x \bullet y + x' \bullet y + x' \bullet y'$

(ii) $x \bullet y \bullet z + x' \bullet y \bullet z + x' \bullet y \bullet z'$

(iii) $x \bullet y \bullet z + x \bullet y' \bullet z' + x' \bullet y \bullet z' + x' \bullet y' \bullet z + x' \bullet y' \bullet z'$

Solution: (i) $x \bullet y + x' \bullet y + x' \bullet y'$

$= (x + x') \bullet y + x' \bullet y'$ (by distributive law)

$= 1 \bullet y + x' \bullet y'$ (inverse property)

$= y + x' \bullet y'$ (identity property)

$= (y + x') \bullet (y + y')$ (by distributive law)

$= (y + x') \bullet 1$ (inverse property)

$= y + x'$ (identity property)

$= y + x' + 0$ (identity property)

$= y + x' + z \bullet z'$ (inverse property)

$= (y + x' + z) \bullet (y + x' + z')$ (by distributive property)

$= (x' + y + z) \bullet (x' + y + z')$ (by commutative property)

which is in CNF.

(ii) $x \bullet y \bullet z + x' \bullet y \bullet z + x' \bullet y \bullet z'$

$= [(x \bullet yz + x' \bullet y \bullet z + x' \bullet y \bullet z')']'$ [as $(x')' = x$]

$= [(x \bullet y \bullet z)' \bullet (x' \bullet y \bullet z)' \bullet (x' \bullet y \bullet z')]'$

$= [(x' + y' + z') \bullet (x + y' + z') \bullet (x + y' + z)]'$ …(1)

= complement of the function $(x' + y' + z')\,(x + y' + z')\,(x + y' + z)$ which is in CNF.

= the function comprising of those factors of the **complete** conjunctive normal form in x, y, z which are not present in this function (1)

$= (x + y + z) \bullet (x' + y + z) \bullet (x + y + z') \bullet (x' + y' + z) \bullet (x' + y + z')$ which is in CNF.

Note: The given function was in DNF and has been changed into CNF.

(iii) $x \bullet y \bullet z + x \bullet y' \bullet z' + x' \bullet y \bullet z' + x' \bullet y' \bullet z + x' \bullet y' \bullet z'$

$= [(x \bullet y \bullet z + x \bullet y' \bullet z' + x' \bullet y \bullet z' + x' \bullet y' \bullet z + x' \bullet y' \bullet z')']'$ [as $(x')' = x$]

$= [(x \bullet y \bullet z)' \bullet (x \bullet y' \bullet z')' \bullet (x' \bullet y \bullet z')' \bullet (x' \bullet y' \bullet z)' \bullet (x' \bullet y' \bullet z')']'$

$= [(x' + y' + z') \bullet (x' + y + z) \bullet (x + y' + z) \bullet (x + y + z') + (x + y + z)]'$

= complement of the function

$(x' + y' + z') \bullet (x' + y + z) \bullet (x + y' + z) \bullet (x + y + z') \bullet (x + y + z)]$ ...(1)

which is in CNF

= the function comprising of those factors of the **complete** conjunctive normal form in x, y, z which are missing in the given function (1)

$= (x' + y' + z) \bullet (x' + y + z') \bullet (x + y' + z')$ which is in CNF.

Example 38: Determine disjunctive normal form equivalent to the following conjunctive normal form:

$$(x + y + z) \bullet (x + y + z') \bullet (x' + y + z) \bullet (x + y' + z') \bullet (x' + y' + z) \bullet (x' + y' + z')$$

Solution: The given expression

$= \{(x + y + z) \bullet (x + y + z')\} \bullet (x' + y + z)\,(x + y' + z')\,(x' + y' + z)\,(x' + y' + z')$

$= (x + y + z \bullet z') \bullet \{(x' + y + z) \bullet (x' + y' + z)\} \bullet \{(x + y' + z')\,(x' + y' + z')\}$

(Combining the first two factors by distributive law and applying commutative law on the rest of factors)

$= (x + y + z \bullet z') \bullet (x' + z + y \bullet y') \bullet (x \bullet x' + y' + z')$

(Combining second and third factors and fourth and fifth factors by distributive law)

$= (x + y + 0) \bullet (x' + z + 0) \bullet (0 + y' + z')$ (by inverse property)

$= (x + y) \bullet (x' + z) \bullet (y' + z')$ (by identity property)

$= (x + y)\,[x' \bullet y' + x' \bullet z' + y' \bullet z + z \bullet z']$ (by distributive and commutative property)

$= (x + y)\,(x' \bullet y' + x' \bullet z' + y' \bullet z + 0)$ (by inverse property)

$= (x + y)\,(x' \bullet y' + x' \bullet z' + y' \bullet z)$ (by identity property)

$= (x \bullet x' \bullet y' + x \bullet x' \bullet z' + x \bullet y' \bullet z) + (x' \bullet y \bullet y' + x' \bullet y \bullet z' + y \bullet y' \bullet z)$

(by distributive and commutative property)

$= 0 \bullet y' + 0 \bullet z' + x\,y'\,z + x' \bullet 0 + x'\,y\,z' + 0 \bullet z$ (by inverse property)

$= 0 + 0 + x\,y'\,z + 0 + x'\,y\,z' + 0$ (by boundedness law)

$= x \bullet y' \bullet z + x' \bullet y \bullet z'$ which is in DNF.

21.1 Reduction of Complete DNF to Identity

Example 39: Convert the following DNF into equivelant CNF by finding values of the function (by using truth table) $xy + x'\,y + x'\,y'$ in 3 variables x, y and z.

Solution: Values of the given function are determined as follows:

Table 1

x	y	z	x′	y′	xy	x′y	x′y′	xy + x′y + x′y′ = f(x, y, z)
0	0	0	1	1	0	0	1	1
0	0	1	1	1	0	0	1	1
0	1	0	1	0	0	1	0	1
0	1	1	1	0	0	1	0	1
1	0	0	0	1	0	0	0	0
1	0	1	0	1	0	0	0	0
1	1	0	0	0	1	0	0	1
1	1	1	0	0	1	0	0	1

In order to convert it into CNF we see that the values of the given expression is 0 in rows 5 and 6.

The values of variables (x, y, z) in row 5 are (1, 0, 0) which, corresponds to the max term $x' + y + z$. The value of variables (x, y, z) in row 6 are (1, 0, 1) which correspond to the max term $x' + y + z'$.

Hence the required CNF is $(x' + y + z) \bullet (x' + y + z')$.

Alternative method:

The given function is

$$f(x, y, z) = x \bullet y + x' \bullet y + x' \bullet y'$$

Values of the function are determined as follows:

Table 2

Row	x	y	z	f (x, y, z) [from table 1
1.	0	0	0	1
2.	0	0	1	1
3.	0	1	0	1
4.	0	1	1	1
5.	1	0	0	0
6.	1	0	1	0
7.	1	1	0	1
8.	1	1	1	1

The three min terms of this function, values of variables (x, y, z) for each min term, and number of row in Table 1 corresponding to these values of variables with its value 1 are given below:

Table 3

Min term	Value of variables (x, y, z) for this min term	Number of row in table 1 corresponding to these value of (x, y, z)	Value of the function in this row
$x \bullet y$	(1, 1, 0) and (1, 1, 1)	7, 8	1
$x' y$	(0, 1, 0) and (0, 1, 1)	3, 4	1
$x' y'$	(0, 0, 0) and (0, 0, 1)	1, 2	1

From this table it is clear that values of the function in each of the remaining i.e. 5^{th} and 6^{th} rows is 0.

The values of the variables (x, y, z) in 5th and 6th rows in which value of the function is zero are (1, 0, 0) and (1, 0, 1) respectively from Table 1.

The max term corresponding to these values are:

$$x' + y + z \text{ and } x' + y + z'$$

The required DNF shall be the product of these max term as given below:

$$(x' + y + z) \bullet (x' + y + z')$$

Example 40: Convert the following CNF into equivalent DNF (by using truth table)

$$(x + y + z) \bullet (x + y + z') \bullet (x' + y + z) \bullet (x + y' + z') \bullet (x' + y' + z) \bullet (x' + y' + z')$$

Solution: The values of the given function are determined as follows: (using method of table-1 of example 39).

Table 1

Row	x	y	z	f (x, y, z)
1.	0	0	0	0
2.	0	0	1	0
3.	0	1	0	1
4.	0	1	1	0
5.	1	0	0	0
6.	1	0	1	1
7.	1	1	0	0
8.	1	1	1	0

We have got eight maxterms in the given function. The different maxterms, values of variables (x, y, z), number of row in table 1 corresponding to these values of (x, y, z) with its value as zero are given below in table 2.

Table 2

Maxterm	Value of variables (x, y, z) for this max term	Number of row in table 1, corresponding to these values of (x, y, z)	Value of the function in this row
$x + y + z$	(0, 0, 0)	1	0
$x + y + z'$	(0, 0, 1)	2	0
$x' + y + z$	(1, 0, 0)	5	0
$x + y' + z'$	(0, 1, 1)	4	0
$x' + y' + z$	(1, 1, 0)	7	0
$x' + y' + z'$	(1, 1, 1)	8	0

From this table it is clear that value of function in each of the remaining i.e. 3^{rd} and 6^{th} rows of table 1 is one.

The values of variables (x, y, z) in 3rd and 6th rows in which value of function is 1 are (0, 1, 0) and (1, 0, 1) from table 1.

The minterms corresponding to these values of x, y and z are $x' \bullet y \bullet z'$ and $x \bullet y' \bullet z$. The required CNF shall be the sum of these minterms as given below

$$x' y \bullet z' + x \bullet y' \bullet z$$

Example 41: Write down a Boolean expression in three variables x, y and z in its complete disjunctive normal (sum of product) form and hence show that it can be reduced to I, the identity for dot $(\bullet)$ or meet operation.

Solution: The required Boolean expression in 3 variables x, y and z in its complete disjunctive normal form can be written as

$$f(x, y, z) = x \bullet y \bullet z + x' \bullet y \bullet z + x \bullet y' \bullet z + x \bullet y \bullet z' + x' \bullet y' \bullet z + x' \bullet y \bullet z' + x \bullet y' \bullet z' + x' y' z'$$

which can be simplified as

$$f(x, y, z) = (x + x') \bullet y \bullet z + x \bullet y' \bullet z + x \bullet y \bullet z' + x' \bullet y' z + x' \bullet y \bullet z' + (x + x') \bullet y' \bullet z'$$ (by distributive law)

$= 1 \bullet y \bullet z + x \bullet y' \bullet z + x \bullet y \bullet z' + x' \bullet y' \bullet z + x' \bullet y \bullet z' + 1 \bullet y' \bullet z'$ (by inverse property)

$= y \bullet z + x \bullet y' \bullet z + x \bullet y \bullet z' + x' \bullet y' \bullet z + x' \bullet y \bullet z' + y' \bullet z'$ (by identity property)

$= y \bullet z + (x \bullet y' \bullet z + x' \bullet y' \bullet z) + x \bullet y \bullet z' + x' \bullet y \bullet z' + y' z'$ (by commutative law)

$= y \bullet z + (x + x') \bullet y' \bullet z + x \bullet y \bullet z' + x' \bullet y \bullet z' + y' \bullet z'$ (by distributive law)

$= y \bullet z + 1 \bullet y' \bullet z + x \bullet y \bullet z' + x' \bullet y \bullet z' + y' \bullet z'$ (inverse property)

$= y \bullet z + y' \bullet z + x \bullet y \bullet z' + x' \bullet y \bullet z' + y' \bullet z'$ (identity property)

$= y \bullet z + y' \bullet z + (x + x') \bullet y \bullet z' + y' \bullet z'$ (by distributive property)

$= y \bullet z + y' \bullet z + y \bullet z' + y' z'$ (inverse and identity property)

$= (y + y') \bullet z + (y + y') \bullet z'$ (by distributive property)

$= 1 \bullet z + 1 \bullet z'$

$= z + z'$

$= 1$ (by inverse property)

22. BOOLEAN FUNCTIONS

We have seen that the last column of logic table of a Boolean expression gives the value of the output Y of circuit represented by the given Boolean expression, for different set of values of the input variables $x_1, x_2 \ldots x_n$. Therefore the output expression $Y = X(x_1, x_2, \ldots x_n)$ defines output values as functions of input bits. This function gives a relation between inputs to the circuit and its outputs.

Let $X(x_1, x_2) = x'_1 \wedge x_2$ be a Boolean expression where x_1 and x_2 can take values in B = {0, 1}. We can calculate the values of this expression for different pairs of values of x_1 and x_2 by using Boolean algebra as represented in the following table

x_1	x_2	x'_1	$x'_1 \wedge x_2$
0	0	1	0
0	1	1	1
1	0	0	0
1	1	0	0

This gives the Boolean function for the expression $(x_1' \wedge x_2)$ which may be denoted as $f : B^2 \to B$ such functions are called **Boolean functions.**

DEF. Let $(B, +, \bullet, ', 0, 1)$ be a Boolean algebra. A function $f : B^n \to B$ defined as

$$f(a_1, a, \ldots, a_n) = E(a_1, a_2, \ldots, a_n) \text{ where } (a_1, a_2, \ldots, a_n) \in B^n \text{ for } a_i \in B$$

is called a Boolean function if it can be specified by Boolean expression of n variables.

Note: Every function $g : B^n \to B$ is not a Boolean function. However, for the case of two-valued Boolean algebra {0, 1} any function from $\{0, 1\}^n$ to {0, 1} is a Boolean function.

Now, for a given Boolean expression, we can find the values of Boolean function as demonstrated in the following example.

Example 42: Let B = {0, 1} and $f : B^3 \to B$ be the function defined by $f(x_1, x_2, x_3) = (x_1 + x_2) \bullet x_3$. Find all the functional values of f.

Solution: We construct the truth table as follows:

Table 1(a)

x_1	x_2	x_3	$x_1 + x_2$	$f \bullet (x_1 + x_2) \bullet x_3$
0	0	0	0	0
0	0	1	0	0
0	1	0	1	0
0	1	1	1	1
1	0	0	1	0
1	0	1	1	1
1	1	0	1	0
1	1	1	1	1

The last column in this table gives the functional values of f e.g., f(0, 0, 0) = 0 and f(0, 0, 1) = 0.

Converse case: If the functional values arc given we want to find the Boolean expression that specifies a given function from $(0, 1)^n$ to (0, 1) then we can obtain the expression in disjunctive normal form or conjunctive normal form as explained below and then simplify it.

If we are given a function from $(0, 1)^n$ to (0, 1) then a Boolean expression in disjunctive normal form corresponding to this function can be obtained by having a minterm corresponding to each ordered n-typle of 0's and 1's for which the values of the function is 1. For each such n-tuple, we have a minterm

$$y_1 \bullet y_2 \dots y_n$$

in which each y_i is x_i if the i^{th} component of the n-tuple is 1 and is x'_i if the ith component of the n-tuple is 0.

Similarly, we can obtain a Boolean expression in conjunctive nomral form (CNF) corresponding to the given function. For each row for which the value of the function is 0, we have a maxterm

$$y_1 + y_2 + \dots + y_n$$

where each y_i is x_i if i^{th} component is 0 and x'_i if i^{th} component of n-typle is 1.

23. EQUIVALENT BOOLEAN EXPRESSIONS

The two Boolean expressions given by $X = X(x_1, x_2, \dots x_n)$ and $Y = Y(x_1, x_2, \dots x_n)$ in n variables are said to be equivalent over the Boolean algebra B = {0, 1}, if both the expressions X and Y define the same Boolean function B.

It means that

$$X(e_1, e_2, \dots, e_n) = Y(e_1, e_2, \dots, e_n) \ \forall \ e_i \in \{0, 1\}$$

For this we find out the values of the two Boolean functions corresponding to two given expressions X and Y. If the values of these functions are identical then the two Boolean expressions are equivalent. This will be more clear form the example given here.

Example 19: Show that the following two Boolean expressions are equivalent over the two element algebra B = {0, 1}

$$X_1 = (x_1 \bullet x_2) + (x_1 \bullet x'_3) \text{ and } X_2 = x_1 \bullet (x_2 + x'_3)$$

Solution: Let f and g be the Boolean functions corresponding to X_1 and X_2 respectively. The values of these two Boolean functions are calculated below in table 1 and 2.

X_1 involves three variables. So the corresponding function f shall be a three variable function i.e. $f : B^3 \to B$ which is defined as

$$f(e_1, e_2, e_3) = (e_1 \bullet e_2) + (e_1 \bullet e'_3),\ e_1, e_2, e_3 \in B.$$

Table 1 Calculation of value of f(e_1, e_2, e_3) or f(x_1, x_2, x_3)

x_1 or e_1	x_2 or e_2	x_3 or e_3	$x_1 \bullet x_2$ or $e_1 \bullet e_2$	x_3' or e_3'	$x_1 \bullet x_3'$ or $e_1 \bullet e_3'$	$(x_1 \bullet x_2) + (x_1 \bullet x_3')$ or $(e_1 \bullet e_2) + (e_1 \bullet e_3')$
1	1	1	1	0	0	1
1	1	0	1	1	1	1
1	0	1	0	0	0	0
1	0	0	0	1	1	1
0	1	1	0	0	0	0
0	1	0	0	1	0	0
0	0	1	0	0	0	0
0	0	0	0	1	0	0

Similarly the value of g is calculated as follows:

Table 2 Calculation of value of g(e_1, e_2, e_3) or g(x_1, x_2, x_3)

x_1 or e_1	x_2 or e_2	x_3 or e_3	x_3' or e_3'	$x_2 + x_3'$ or $e_2 + e_3'$	$x_1 \bullet (x_2 + x_3')$ or $e_1 \bullet (e_2 + e_3')$
1	1	1	0	1	1
1	1	0	1	1	1
1	0	1	0	0	0
1	0	0	1	1	1
0	1	1	0	1	0
0	1	0	1	1	0
0	0	1	0	0	0
0	0	0	1	1	0

As the last columns of the two tables are identical, the two expressions x_1 and x_2 are equivalent.

24. VALUE OF BOOLEAN EXPRESSION

Suppose that E (x_1, x_2,…, x_n) be a Boolean expression of n variables over a Boolean algebra $(B, \vee, \wedge, ', 0, 1)$ or $(B, +, \bullet, ', 0, 1)$ and let B = {a_1, a_2, …a_n}. Let (a_1, a_2, …a_n) which is an n-tuple. If we replace x_1 by a_1, x_2 by a_2, … and x_n by a_n in the Boolean expression E (x_1, x_2,…, x_n), we obtain an expression which is an element of B. The expression E (a_1, a_2,…,a_n) $\in$ B is called the value of the Boolean expression E (x_1, x_2,…, x_n) for the n-tuple (a_1, a_2,…, a_n) $\in$ B^n. The value of Boolean expression E (x_1, x_2,…, x_n) can be determined for every n-tuple (a_1, a_2,…, a_n) $\in$ B^n.

For example, for the Boolean expression

$$E(x_1, x_2, x_3) = (x_1 + x_2)(x_1' + x_2')(x_2 + x_3)'$$

over the Boolean algebra {(0, 1), +, . ,′} the assignment of values $x_1 = 0, x_2 = 1, x_3 = 0$ yields value of the Boolean expression as

$$E(0, 1, 0) = (0 + 1)(0' + 1')(1 + 0)'$$
$$= 1 \bullet 1 \bullet 0$$
$$= 0.$$

Example 44: Prove that: $(x_1 + x_2) \bullet (x_1' + x_3) \equiv (x_1 \bullet x_3) + (x_1' \bullet x_2)$.

Solution: We will show the equivalence of two expressions over the Boolean algebra (0, 1) by evaluating $(x_1 + x_2) \bullet (x_1' + x_3)$ and $(x_1 \bullet x_3) + (x_1' \bullet x_2)$ for each of the eight possible assignments to the variables x_1, x_2, x_3 as shown below:

x_1	x_2	x_3	x_1'	$x_1 + x_2$	$x_1' + x_3$	$x_1 \bullet x_3$	$x_1' \bullet x_2$	$(x_1 + x_2) \bullet (x_1' + x_3)$	$(x_1 \bullet x_3) + (x_1' \bullet x_2)$
0	0	0	1	0	1	0	0	0	0
0	0	1	1	0	1	0	0	0	0
0	1	0	1	1	1	0	1	1	1
0	1	1	1	1	1	0	1	1	1
1	0	0	0	1	0	0	0	0	0
1	0	1	0	1	1	1	0	1	1
1	1	0	0	1	0	0	0	0	0
1	1	1	0	1	1	1	0	1	1

Similarly of the last two columns show that the expression on R.H.S is equivalent to the expression on the L.H.S.

Example 45: Construct the truth table for the Boolean function $f : B_3 \to B$ determined by the Boolean polynomial or Boolean expression

$$p(x, y, z) = (x \wedge y) \vee (x \vee (y' \wedge z))$$

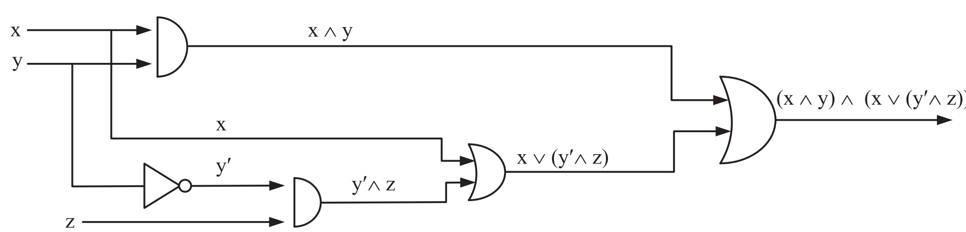
or $$p(x, y, z) = (x \bullet y) + (x + (y' \bullet z))$$ where $B = \{0, 1\}$.

Also draw the logic diagram for this expression.

Solution:

x	y	z	y	$(x \wedge y) \vee (x \vee (y' \wedge z))$
0	0	0	1	$(0 \wedge 0) \vee (0 \vee (1 \wedge 0)) = 0 \vee (0 \vee 0) = 0 \vee 0 = 0$
0	0	1	1	$(0 \wedge 0) \vee (0 \vee (1 \wedge 1)) = 0 \vee (0 \vee 1) = 0 \vee 1 = 1$
0	1	0	0	$(0 \wedge 1) \vee (0 \vee (1 \wedge 0)) = 0 \vee (0 \vee 0) = 0 \vee 0 = 0$
0	1	1	0	$(0 \wedge 1) \vee (0 \vee (0 \wedge 1)) = 0 \vee (0 \vee 0) = 0 \vee 0 = 0$
1	0	0	1	$(1 \wedge 0) \vee (1 \vee (1 \wedge 0)) = 0 \vee (1 \vee 0) = 0 \vee 1 = 1$
1	0	1	1	$(1 \wedge 0) \vee (1 \vee (1 \wedge 1)) = 0 \vee (1 \vee 1) = 0 \vee 1 = 1$
1	1	0	0	$(1 \wedge 1) \vee (1 \vee (0 \wedge 0)) = 1 \vee (1 \vee 0) = 1 \vee 1 = 1$
1	1	1	0	$(1 \wedge 1) \vee (1 \vee (0 \wedge 1)) = 1 \vee (1 \vee 0) = 1 \vee 1 = 1$

Its logic diagram is given below:



Example 46: Use truth table to verify the validity of the distribution law

$$x \bullet (y + z) \equiv x \bullet y + x\,z$$

in Boolean algebra.

Solution: The table is shown below:

x	y	z	y + z	xy	xz	x • (y + z)	x • y + x • z
0	0	0	0	0	0	0	0
0	0	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	1	1	1	0	0	0	0
1	0	0	0	0	0	0	0
1	0	1	1	0	1	1	1
1	1	0	1	1	0	1	1
1	1	1	1	1	1	1	1

Agreement of the last two columns establish the validity of the given identity.

Example 47: Construct the function f from the following table:

x	y	z	f	T
1	1	1	0	$x' y' z'$
1	1	0	1	$x' y' z$
1	0	1	1	$x' y z'$
1	0	0	1	$x' yz$
0	1	1	0	$x y' z'$
0	1	0	0	$x y' z$
0	0	1	1	$x y z'$
0	0	0	0	$x y z$

where T stands for the term of the function. **(UPTU., B.Tech. 2003)**

Solution: Using Boole's expansion theorem, we have

$$f = 0\,(x' y' z') + 1\,(x' y' z) + 1\,(x' y z') + 1\,(x' y z) + 0\,(x y' z') + 0\,(x y' z) + 1\,(x y z') + 0\,(x y z)$$

$$= x' y' z + x' y z' + x' y z + x y z'$$

$$= x' y' z + x' y (z' + z) + x y z' \qquad \text{(by distributive law)}$$

$$= x' y' z + x' y \bullet 1 + x y z' \qquad \text{(by identity property } \because z + z' = 1)$$

$$= x' y' z + x' y + x y z'$$

$$= x' y' z + x' y + x y z'$$

$$= x' y' z + (x' + xz') y \qquad \text{(by distributive law)}$$

$$= x' y' z + y (x' + xz')$$

$$= x' y' z + y (x' + x)(x' + z') \qquad \text{(by distributive law)}$$

$= x' y' z + y \bullet 1 (x' + z')$ (as $x' + x = 1$)

$= x' y' z + y (x' + z')$

$= x' y' z + yx' + yz'$

$= x' y' z + x' y + yz'$

$= x' (y' z + y) + y z'$

$= x' (y' + y) (z + y) + yz'$

$= x' \bullet 1 (z + y) + yz'$

$= x' (y + z) + yz'$

$= x' y + x' z + yz'$

25. MINIMAL BOOLEAN FUNCTION

A Boolean function in n variable $x_1, x_2, \ldots, x_n$ is said to be minimal if it is the product of n variables provided the r^{th} variable is either taken x_r or its complement x'_r. For example: Let x and y be two variable in Boolean Algebra B and the complements of x and y be x′ and y′. Then the minimal Boolean functions are given by

$x \bullet y, x' \bullet y, x \bullet y', x \bullet y'$

From above minimal Boolean functions, we conclude that the number of minimal Boolean function in two variables is $2^2 = 4$.

Similarly, we consider three variables in a Boolean Algebra, then the minimal Boolean function are given by

$x \bullet y \bullet z, x' \bullet y \bullet z, x \bullet y' \bullet z, x \bullet y \bullet z', x' \bullet y' \bullet z, x' \bullet y \bullet z', x \bullet y' \bullet z', x' \bullet y' \bullet z'$

Thus the number of minimial Boolean functions in three variables is $2^3 = 8$.

Bool's Theorem Statement: There are 2^n *minimal Boolean functions* in n variables.

Proof: Let $x_1, x_2, x_3, \ldots, x_n$ be n variables in a Boolean Algebra B and let $x'_1, x'_2, x'_3, \ldots, x'_n$ be the complements of the above variables respectively. To form a minimal Boolean function each variable can be selected in two ways, that is either x_r is taken or x'_r. Since there are n variables, thus the number of minimal Boolean function are

$$2 \times 2 \times 2 \times \ldots \times \text{upto n times} = 2^n$$

Hence the number of minimal Boolean function = 2^n.

Example 48: Obtain the Boolean expression, of the functions f (x, y, z) whose truth table are given below:

(i)

x	y	z	f (x, y, z)
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	0

(ii)

x	y	z	f (x, y, z)
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	0

(ii)

x	y	z	f (x, y, z)
0	0	0	1
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1

[UPTU., MCA-IV Sem 2002-03]

Solution: (i) The value of f (x, y, z) is 1 in sixth row in which the values of variables x, y and z are 1, 0 and 1 respectively. These values shall corrrespond to the minterm $x\, y'\, z$.

(ii) The value of f (x, y, z) is 1 in third and seventh rows Values of x, y and z in third row are 0, 1 and 0 which correspond to the minterm $x' \bullet y \bullet z'$.

Values of x, y and z in seventh row are 1, 1 and 0 which corresponds to the minterm $x \bullet y \bullet z'$.

Hence, the rquired function shall be the sum of these two minterm i.e.

$$f\,(x, y \bullet z) = x'\, y\, z' + x \bullet y \bullet z'$$

(iii) The value of the function f (x, y, z) is 1 in 1st, 2nd, 5th and 8th rows.

The values of variables x, y and z, in 1st row are 0, 0 and 0 which correspond to minterm $x' \bullet y' \bullet z'$.

The values of x, y and z in 2nd row are 0, 0 and 1 which correspond to minterm $x' \bullet y' \bullet z$.

The values of x, y and z in 5th row are 1, 0 and 0 which correspond to minterm $x \bullet y' \bullet z'$.

The values of x, y and z in 8th row are 1, 1 and 1 which correspond to minterm $x \bullet y \bullet z$.

The required function shall be the sum of all these minterm i.e.

$$f\,(x, y, z) = x' \bullet y' \bullet z' + x' \bullet y' \bullet z + x \bullet y' \bullet z' + x \bullet y \bullet z$$

Note: The Boolean expressions determined are in DNF.

In case we want to determine the Boolean expressions corresponding to the given truth tables of the functions, we shall consider the rows in which value of the expression is 0 and find the maxterm corresponding to such rows. The required Boolean expression in CNF shall be the product of all these maxterm as shown below.

(i)

x	y	z	f (x, y, z)	Maxterm corresponding to zero value of f
0	0	0	0	$x + y + z$
0	0	1	0	$x + y + z'$
0	1	0	0	$x + y' + z$
0	1	1	0	$x + y' + z'$
1	0	0	0	$x' + y + z$
1	0	1	1	— — — —
1	1	0	0	$x' + y' + z$
1	1	1	1	— — — —

Required function in CNF is

$$(x + y + z) \bullet (x + y + z') \bullet (x + y' + z) \bullet (x + y' + z') \bullet (x' + y + z) \bullet (x' + y' + z)$$

(ii)

x	y	z	f (x, y, z)	Max term corresponding to zero value of f
0	0	0	0	$x + y + z$
0	0	1	0	$x + y + z'$
0	1	0	1	$x + y' + z$
0	1	1	0	$x + y' + z'$
1	0	0	0	$x' + y + z$
1	0	1	0	$x' + y + z'$
1	1	0	1	$x' + y' + z$
1	1	1	0	$x' + y' + z'$

Required function in CNF is

$$(x + y + z) \bullet (x + y + z') \bullet (x + y' + z') \bullet (x' + y + z) \bullet (x' + y + z') \bullet (x' + y' + z')$$

(iii)

x	y	z	f (x, y, z)	Max term corresponding to zero value of f
0	0	0	1	— — — —
0	0	1	1	— — — —
0	1	0	0	$x + y' + z$
0	1	1	0	$x + y' + z'$
1	0	0	1	— — — —
1	0	1	0	$x' + y + z'$
1	1	0	0	$x' + y' + z$
1	1	1	1	— — — —

Required CNF is

$$(x + y' + z) \bullet (x + y' + z') \bullet (x' + y + z') \bullet (x' + y' + z)$$

Note 2: For the sake of convenience only, we determine the required Boolean expression in DNF if the number of 1s in the column of values of the function is less than 0s and in CNF if the number of 0s in the column of values of the function is less than the number of 1s.

Example 49: Find value of the Boolean function given by

$$f(x, y, z) = x\, y + z'$$

Solution: We know that each one of these variables can take the value 0 or!

All possible values of the given function are shown in the following table:

x	y	z	$x \bullet y$	z'	$x \bullet y + z'$
0	0	0	0	1	1
0	0	1	0	0	0
0	1	0	0	1	1
0	1	1	0	0	0
1	0	0	0	1	1
1	0	1	0	0	0
1	1	0	1	1	1
1	1	1	1	0	1

26. MINIMAL SUM OF PRODUCTS

Consider a Boolean sum-of-product expression E

E_L= number of literals in E (Counted according to multiplicity)

E_S = number of summands in E

Illustration: Let E = xyz′ + x′y′t + xy′z′t + x′yzt

Then $E_L = 3 + 3 + 4 + 4 = 14$, $E_S = 4$

Suppose E and F are equivalent. Boolean sum-of-products expression, we say

E is simpler than F, if

(i) $E_L < F_L$ and (ii) $E_S \le F_S$

We say E is minimal if there is no equivalent sum of products expression which is simpler than E. There can be more then one equivalent minimal sum of products expressions.

27. PRIME IMPLICANTS

A fundamental product P is called a prime implicant of a Boolean expression E if P + E = E but no other fundamental product contained in P has this property.

Example: Let E = xy′ + xyz′ + x′yz′

We can show that xz′ + E = E but x + E ≠ E and z′ + E ≠ E. Thus xz′ is a prime implicant of E. Hence A minimal sum of products form for a Boolean expression E is a sum of prime implicants of E.

28. CONSENSUS OF FUNDAMENTAL PRODUCTS

P_1 and P_2 are fundamental products such that exactly one variable, say x_k appears uncomplemented in one of P_1 & P_2 and complemented in the other, then the consensus of P_1 and P_2 is the product (without repetitions) of the literals of P_1 and the literals of P_2 after x_k and x'_k are deleted. If Q is consensus of P_1 and P_2, then $P_1 + P_2 + Q = P_1 + P_2$.

How to find Consensus

Example 49: Find Consensus Q of P_1 and P_2 where

(a) P_1 = xyz′s and P_2 = xy′t (b) P_1 = xy′ and P_2 = y

(c) P_1 = x′yz, P_2 = x′yt (d) P_1 = x′yz and P_2 = xyz′

Solution:

(a) Deleting y and y′ & then multiplying literals of P_1 and P_2 (without repetitions) we get
Q = xz′st

(b) P_1 = xy′ and P_2 = y
Deleting y and y′ gives
Q = x.

(c) P_1 = x′yz, P_2 = x′yt
P_1 & P_2 have no consensus as no variable appears as an uncomplemented in one of the products and complemented in the other.

(d) P_1 = x′yz and P_2 = xyz′
Each of x and z (more than one & not exactly one) appear complemented in one of the product and uncomplemented in the other)
Hence P_1 and P_2 have no consensus.

How to find Prime implicants of a Boolean Expression—Algorithm

Input : Boolean expression $E = P_1 + P_2 + \dots + P_n$ where P's are fundamental products.

Output: Expressed as sum of its prime implicants.

Step-I Delete any fundamental product P_i which includes any other fundamental product P_j (permissible by absorption law).

Step-II Add consensus of any P_i and P_j provided Q does not include any of the Ps (permissible by $P_1 + P_2 + Q = P_1 + P_2$).

Step-III Repeat step I and/or step II.

29. SWITCHING CIRCUITS

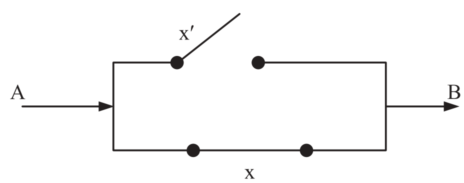
A switch in an electric circuit is a device which has two basic states. One state is the **closed state** called its **on position** when it allows the current to flow in the circuit. The other state is **open state** called its **off position** when it does not allow the current to flow in the circuit.

We denote the switch in circuit by x and assign value 0 to x when the switch is in open state or off position and assign a value 1 to x when the switch is in closed state or on position. These values 0 and 1 which denote the two states of the switch are called **state-values** of the switch.

We also use x' or x^c to denote the switch whose state is opposite to that of the switch x. It means that if x denotes a switch in on position then x' denotes a switch in off position and vice versa. Switch x' is called invert of x.

The state values of x and x' are given below in tabular form and their positions shown in the diagram-

State values of	
x	x'
0	1
1	0



Since the variable x denoting the switch can assume two values 0 and 1, it is called a Boolean variable.

29.1 Two ways of connecting two switch in a circuit

1. Parallel Connections: In this type of connection the switches are connected in such a way that the current will flow from one end to the other only when at least one of the two switches are in a closed state or on-position i.e. the values of at least one of the two switches (say x_1 and x_2) is equal to 1. ($x_1 = 1$, or $x_2 = 1$ or $x_1 = x_2 = 1$). In such a case the state value of the circuit connection is 1 when the current flows and 0 when the current does not flow through it. This can be depicted in tabular forms as given below for two switches x_1 and x_2

State value of		
x_1	x_2	Circuit connection (x_1 parallel x_2)
0	0	0
0	1	1
1	0	1
1	1	1

2. Series Connection: In this type of connection the switches are connected in such a way that the current will flow from one end to the other end of the circuit when both the switches are in closed state or on-position (i.e. the value of x for the two switches (say x_1 and x_2) each is equal to 1 (x_1 == 1, x_2 = 1). The state value of the circuit connection in this type of connection can also be shown in tabular form as given below:

State value of		
x_1	x_2	Circuit connection (x_1 in series with x_2)
0	0	0
0	1	0
1	0	0
1	1	1

29.2 Switching circuit and Boolean Algebra

Let 0 and 1 denote the two states of a switch as well as the circuit connection (i.e. x = 0, 1). Let S = {0, 1} be a set. Let the two operation of connecting the switches in series and parallel be denoted by $\wedge$ and $\vee$ respectively and the operation of inversion be denoted by / (x' is invert of x). Then (S, parallel, series, /, 0, 1) or (S, $\vee$, $\wedge$, /, 0, 1) is a **Boolean algebra** where S = {0, 1}, 0 and 1 are the identities for the operations of parallel ($\vee$) and series ($\wedge$) connections. It should be noted here that 0 and 1 denotes the two states (off and on) of a switch and has nothing to do with the numbers 0 and 1.

29.3 We shall prove that (S, $\vee$, $\wedge$, / , 0, 1) is a Boolean algebra

1. If we connect any two switches in any of the two states (0 and 1) either in series ($\wedge$) or in parallel ($\vee$) then the resulting state of the circuit shall also be any one of the two states (0 and 1) i.e. the result of any of the two operations ($\wedge$ or $\vee$) on any two of the elements of the set S = {0, 1} is again an element belonging to S. Thus the two operations of circuit connection in series ($\wedge$) and parallel ($\vee$) are closed for the elements of S.

2. If x_1 and x_2 are two switches in any one of two states (0 & 1) connected in parallel ($\vee$) or series ($\wedge$) the resulting state (0 or 1) of the circuit remains unchanged on interchanging the states of x_1 and x_2. Thus the two operations of series ($\wedge$) and parallel ($\vee$) are commutative.

3. If we have three switches x_1, x_2 and x_3 then the result of order of application of the operation $\vee$ or $\wedge$ does not affect the result i.e.

$$x_1 \vee (x_2 \vee x_3) = (x_1 \vee x_2) \vee x_3$$

and
$$x_1 \wedge (x_2 \wedge x_3) = (x_1 \wedge x_2) \wedge x_3$$

It means the two operations are associative.

4. It can also be shown that the each of the two operations distributes over the other i.e.

$$x_1 \vee (x_2 \wedge x_3) = (x_1 \vee x_2) \wedge (x_1 \vee x_3) \qquad \ldots(1)$$

and
$$x_1 \wedge (x_2 \vee x_3) = (x_1 \wedge x_2) \vee (x_1 \wedge x_3) \qquad \ldots(2)$$

Thus is easily seen with the help of the following table giving the results of operations.

x_1 (1)	x_2 (2)	x_3 (3)	$x_2 \wedge x_3$ (4)	$x_1 \vee (x_2 \wedge x_3)$ (5)	$x_1 \vee x_2$ (6)	$x_1 \vee x_3$ (7)	$(x_1 \vee x_2) \wedge (x_1 \vee x_3)$ (8)
0	0	0	0	0	0	0	0
0	0	1	0	0	0	1	0
0	1	0	0	0	1	0	0
0	1	1	1	1	1	1	1
1	0	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	1	0	0	1	1	1	1
1	1	1	1	1	1	1	1

Hence columns (5) and (8) are identical, which proves the distributive property (1). Similarly we can prove the distributive property (2).

5. If we connect a switch x_1 in state 0 or 1 in parallel with state 0 of the second switch x_2, the resulting state is 0 or 1 respectively. The same is the result if we interchange x_1 & x_2 i.e.

$$0 \vee 0 = 0 = 0 \vee 0$$

$$1 \vee 0 = 1 = 0 \vee 1$$

It means 0 is the identity for the operation 'parallel' ($\vee$) Similarly if we connect x_1 in state 0 or 1in series with x_2 in state 1, the resulting state of the circuit is 0 or 1 respectively. The result remains the same if we interchange x_1 and x_2

i.e. $$0 \wedge 1 = 0 = 1 \wedge 0$$

$$1 \wedge 1 = 1 = 1 \wedge 0$$

Thus the 1 is the identity for the operation 'series' ($\wedge$)

6. If we connect any one switch (in state 0) with the other switch (in state 1) in parallel we get state 1 (the identity for series) (i.e. $1 \vee 0 = 1 = 0 \vee 1$) . Similarly if we connect any one switch (in state 0) with the other switch (in state 1) in series, we get the state 0 for the circuit (which is identity for parallel). Thus 0 is the complement of 1 and 1 is complement of 0.

Hence (S, $\wedge$, $\vee$, /, 0 , 1) is a Boolean algebra.

29.4 Aims of Boolean algebra of switching circuits

1. Representation of switching circuits by Boolean expressions.
2. Representation of Boolean expression through a switching circuit.

The ultimate aim is the reduction of circuits into simpler form.

Any one circuit is said to be in a simpler form as compared to that of the other if the first circuit contains less number of switches than the other.

30. REPRESENTATION OF BOOLEAN EXPRESSIONS THROUGH SWITCHING CIRCUITS

Here • or $\wedge$ shall denote a connection of switches in series and + or $\vee$ shall represent the connection of switches in parallel. The symbol / denotes inversion or complementation i.e. if x_1 denotes on state then x_1' shall denote off state of the switch x_1 and vice versa.

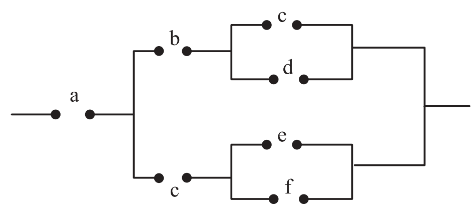
Example 50: Draw the circuit represented by the polynomial.

$$a\,[b\,(c + d) + c\,(e + f\,)]$$

Solution: The required circuit can be drawn in the following steps.

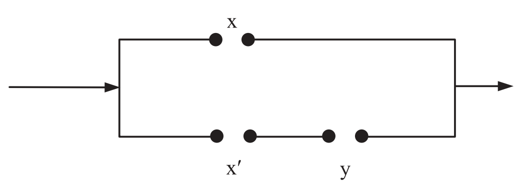
1. This circuit contains a switch a in series with the combination b (c + d) + c (e + f)
2. The combination b (c + d) + c (e + f) contains the two sub-circuits b (c + d) and c (e + f) connected in parallel to each other in the circuit.
3. b (c + d) contains the switch b connected in series with combination of c and d in parallel.
4. c (e + f) contains the switch c connected in series with combination of swtiches e and f in parallel.

   Combining all the four parts the circuit in as shown below:

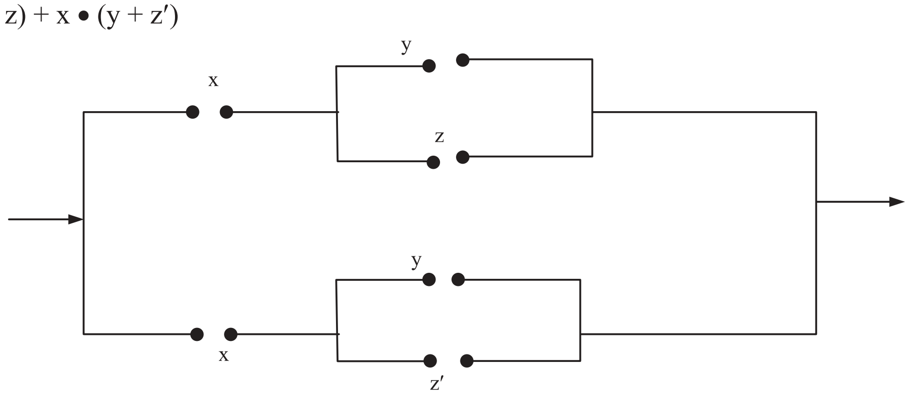


1. Various Boolean expressions and their corresponding switching circuits are given below:

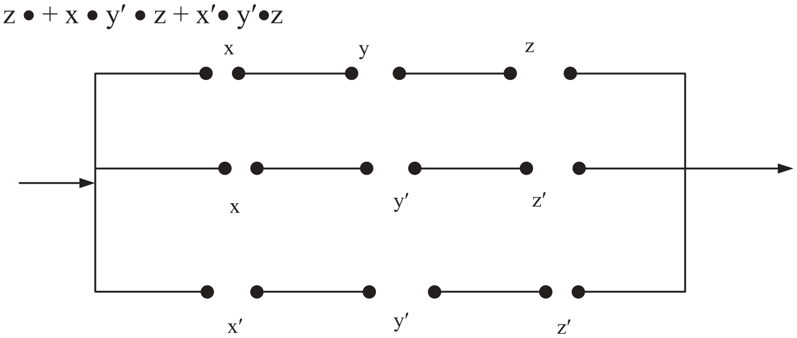
(1) x + x′ • y



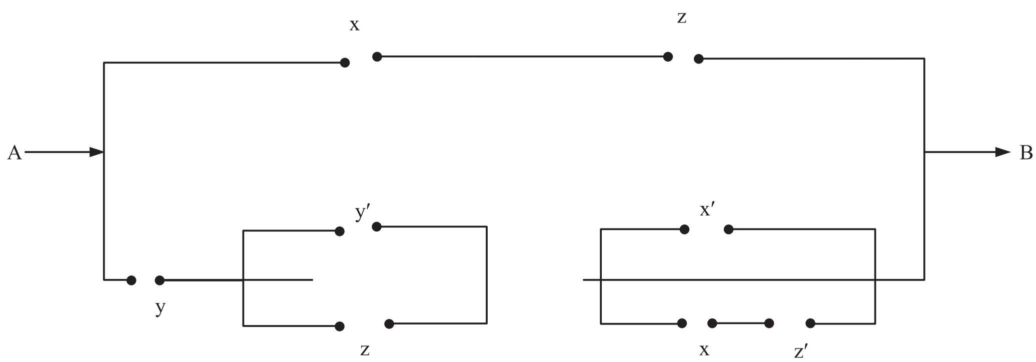
(2) x • (y + z) + x • (y + z′)



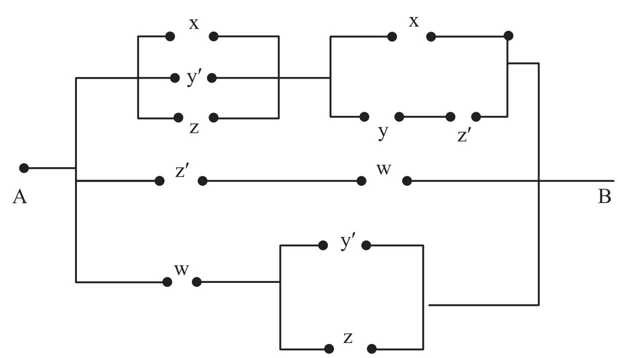
(3) x • y • z • + x • y′ • z + x′• y′•z



(4) $x \bullet z + [y \bullet (y' + z) \bullet (x' + x \bullet z')$



(5) $(x + y' + z)\,(x + y\,z') + z'w + w\,(y' + z)$



Note: 1 If the circuit is given we can find out the corresponding Boolean expression also.

2. Simplified circuits of Boolean expressions: In such cases the given Boolean expression is first simplified by using laws of Boolean algebra and then the switching circuit corresponding to the simplified Boolean expression is drawn.

Example 51: Draw the simplified circuit of the Boolean expression

$a{\bullet}b{\bullet}c + a{\bullet}b'{\bullet}c + a'{\bullet}b'{\bullet}c$ and test the equivalence of two circuits.

Solution: First we simplify the given expression as follows:

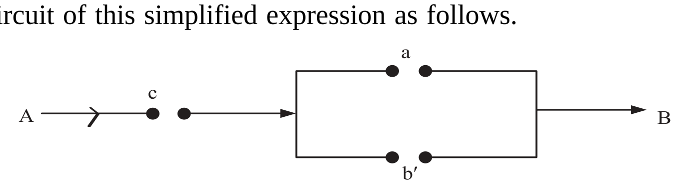
$$a{\bullet}b{\bullet}c + a{\bullet}b'{\bullet}c + a'{\bullet}b'{\bullet}c = (ab + a{\bullet}b' + a'{\bullet}b'){\bullet}c \quad \ldots(1)$$

$$= c{\bullet}[a{\bullet}(b + b') + a'{\bullet}b'] = c{\bullet}[a{\bullet}1 + a'{\bullet}b']$$

$$= c{\bullet}[a + a'{\bullet}b'] = c{\bullet}[(a + a'){\bullet}(a + b')]$$

$$= c{\bullet}1{\bullet}(a + b') = c{\bullet}(a + b') \quad \ldots(2)$$

and then draw the circuit of this simplified expression as follows.



The equivalence of the two circuits is verified by finding the truth values of the two expressions (1) and (2) representing the two circuits as given below:

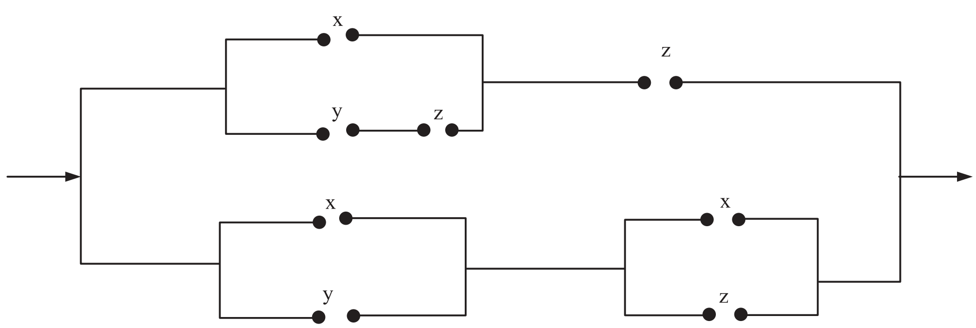
(a) (1)	(b) (2)	(c) (3)	a′ (4)	b′ (5)	a • b • c (6)	a • b′• c (7)	a′ • b′ • c (8)	a b c + a b′c + a′ • b′ • c (9)	a + b′ (10)	c • (a + b′) (11)
0	0	0	1	1	0	0	0	0	1	0
0	0	1	1	1	0	0	1	1	1	1
0	1	0	1	0	0	0	0	0	0	0
0	1	1	1	0	0	0	0	0	0	0
1	0	0	0	1	0	0	0	0	1	0
1	0	1	0	1	0	1	0	1	1	1
1	1	0	0	0	0	0	0	0	1	0
1	1	1	0	0	1	0	0	1	1	1

We see that the truth values of col (9) and col (11) representing expression (1) and (2) respectively are the same. Hence the two expressions as well as the circuits represented by them are equivalent.

31. REPRESENTATION OF A GIVEN CIRCUIT INTO A SIMPLIFIED FORM

This is done by first converting the given circuit into a Boolean expression and then simplifying this Boolean expression by using laws of Boolean algebra. The simplified Boolean expression thus obtained is finally represented by a circuit.

Example 52: Represent the following circuit into a simplified form



Solution: The given circuit shall be represented by the Boolean expression given below

$(x + y•z)•z + (x + y)•(x + z)$

which if simplified becomes

$x•z + (y•z)•z + x + y•z$

$= x•z + y•z + x + y•z$ (by associative and idempotent laws)

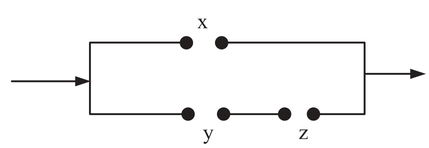
$= x•z + (y•z + y•z) + x$

$= x•z + y•z + x$ (as a + a = a)

$= x•z + x•1 + y•z = x•(z + 1) + y•z$

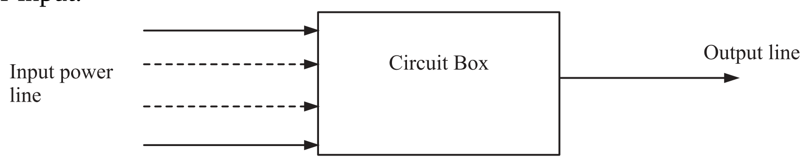
$= x•1 + y•z = x+ y•z$

whose circuit is drawn here.



32. LOGIC GATES AND CIRCUITS

Logic circuits are structures made of certain elementary circuits called logic gates. It is a machine containing one or more input devices and only one output device. Each input device sends a signal in binary digits 0 and 1. The following figure shows a box which consists of a number of electric switches or logic gates, wired together in some specified way. Each line entering the box form left represents an independent power source called input (out of which some or all lines may supply voltage to the box at a particular time). A single line coming out of the box gives the final output which depends on the nature of input.



Thus a gate may be considered as on or off according to whether the output level is 1 or 0 respectively.

33. TYPES OF LOGIC GATES

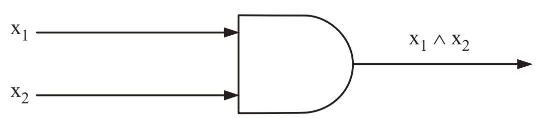
1. AND – gate: Let x_1 and x_2 be the Boolean variables (each having value 0 or 1) representing the two inputs. An AND-gate receives two inputs x_1 and x_2 to give an out-put denoted by $x_1 \wedge x_2$ or $x_1 \bullet x_2$ whose value depends upon the values of x_1 and x_2 both. The nature of the output for inputs x_1 and x_2 in AND-gate can be represented in tabular form as given below.

Logic Table for AND-gate

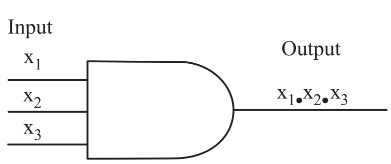
x_1	x_2	$x_1 \wedge x_2$ or $x_1 \bullet x_2$
0	0	0
0	1	0
1	0	0
1	1	1

It is clear from this table that output voltage of the gate is 1 only when the input voltage of each of the two inputs is 1. it is zero otherwise.

The standard pictorial representation of an AND-gate is shown below:



The diagram along with its truth table for three input AND-gate is given below:



x_1	x_2	x_3	$x_1 \bullet x_2 \bullet x_3$
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	0
1	1	1	1

Illustration: Let $x_1 = 1110$, $x_2 = 0111$, $x_3 = 0101$ be input sequences for the AND gate.

The AND gate yields 1 only when all input bits are 1. This occurs only in 2nd position. Thus the output sequence is $x_1 \bullet x_2 \bullet x_3 = 0100$.

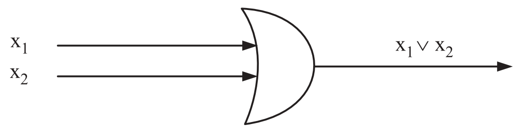
2. OR-gate: If x_1 and x_2 denote the two inputs then the output of such a gate is represented by $x_1 \vee x_2$ or $x_1 + x_2$ whose value (0 or 1) depend upon the values of the inputs x_1 and x_2 as shown in the following logic table

Logic Table for OR-gate

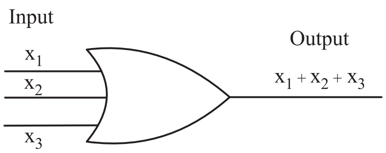
x_1	x_2	$x_1 \vee x_2$ or $x_1 + x_2$
0	0	0
0	1	1
1	0	1
1	1	1

It is clear from this table that the output voltage of an OR-gate is at level 1 whenever the level of any one or both of the inputs wires is 1.

The standard diagrammatic representation of OR-gate is shown below



A diagram along with its truth table for three input OR-gate is given below:



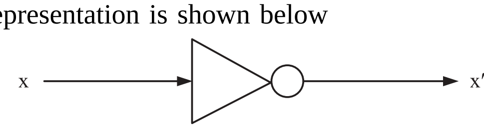
x_1	x_2	x_3	$x_1 + x_2 + x_3$
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

Example: Let x_1 = 1010, x_2 = 0100 and x_3 = 1100 be the input sequences for three input OR-gate then the output sequence will be f (x_1, x_2, x_3) = 1110. The output bit is 0 only when all input bits are 0.

3. NOT-gate: It is such a type of gate that receives an input x (whose value may be 0 or 1) and produces an output denoted by x' (whose value shall be 1 or 0 according to as the value of x is 0 or 1 respectively) as shown in the following logic table.

x	x'
0	1
1	0

Its standard diagrammatic representation is shown below



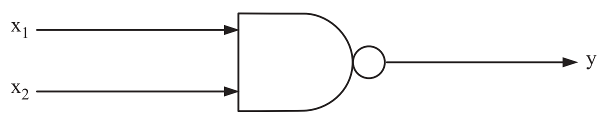
A NOT gate can have only one input, whereas the OR and AND gates may have two or more inputs. All these gates are elementary gates. We can design a logic circuit by using different combinations of these elementary gates in which output of any of these gates is used as input of the other gate.

4. NAND gate: This gate is equivalent to a combination of AND gate followed by a NOT gate. A NAND-gate receives two inputs x_1 and x_2 to give an output denoted by $(x_1 \wedge x_2)'$ or $(x_1 \bullet x_2)'$ whose value depends upon the values of x_1 and x_2. The nature of output for input x_1 and x_2 in this gate can be represented in tabular form as given below.

Logic Table for NAND-gate

x_1	x_2	NAND $(x_1 \bullet x_2)'$
0	0	1
0	1	1
1	0	1
1	1	0

It's standard representation is just like that of AND gate followed by a circle as shown below:

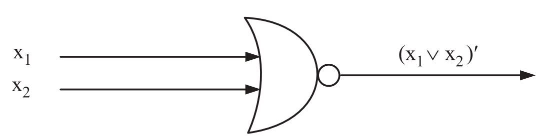


5. NOR gate: It is equivalent to an OR-gate followed by a NOT gate. The tabular representation of the output value y of two inputs x_1 and x_2 to a NOR-gate is shown below:

Logic Table for Nor-gate

x_1	x_2	NOR $(x_1 + x_2)'$
0	0	1
0	1	0
1	0	0
1	1	0

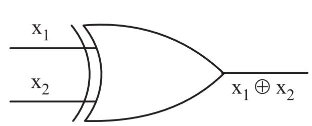
Its standard representation is just like that of OR-gate followed by a circle as shown below:



Exclusive-OR (XOR) gate: It is different from an OR gate as it includes only input sequences that have an odd number of 1's. XOR gate for two inputs x_1 and x_2 is represented as

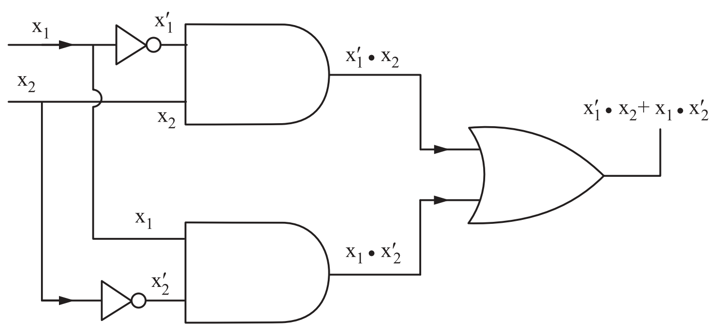
$$f(x_1, x_2) = x_1 \oplus x_2 = x_1' \bullet x_2 + x_1' \bullet x_2'$$

The diagram and its truth table for two input XOR-gate is given below:



x_1	x_2	Output $f(x_1, x_2) = x_1' \bullet x_2 + x_1 \bullet x_2'$
0	0	1
0	1	0
1	0	0
1	1	0

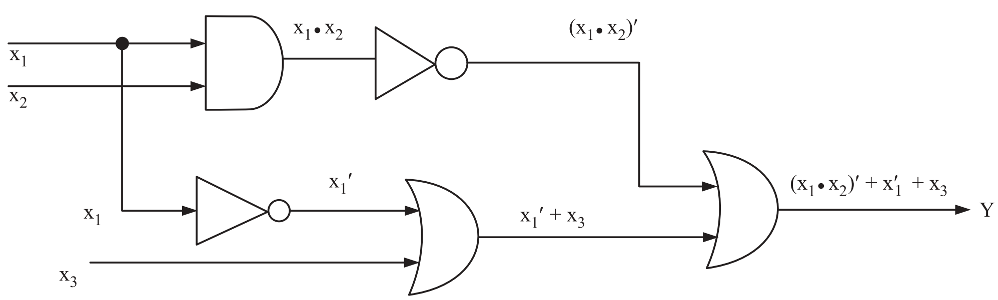
The logic circuit of two input XOR gates can be determined as shown below:



Tow-input XOR gate

34. REPRESENTATION OF LOGIC CIRCUITS BY BOOLEAN EXPRESSIONS

In the following logic circuit the inputs are x_1, x_2 and x_3. Output is y.



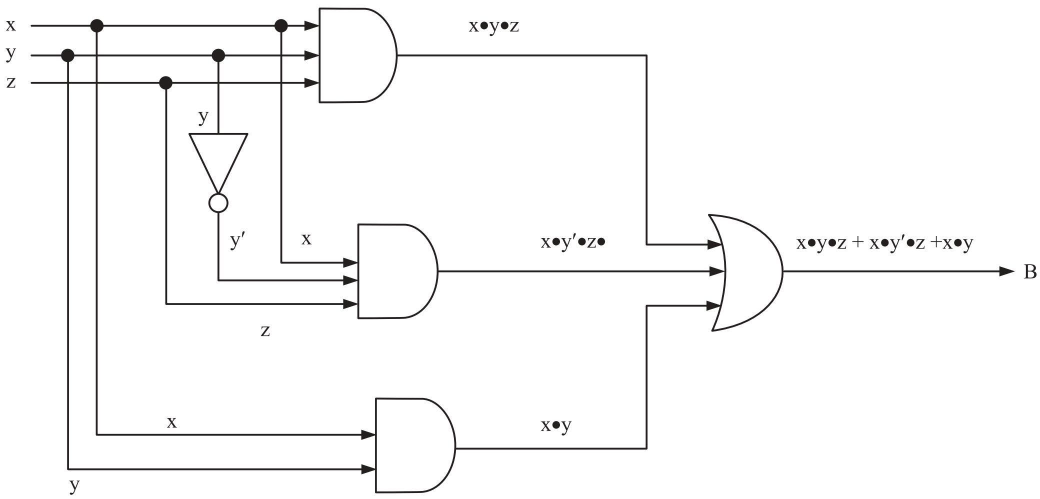
A dot (as in the input line of x_1 indicates the point where input line splits and its bit signal is sent in more than one direction.

Hence we see that the inputs x_1 and x_2 are converted by AND-gate into an output $x_1 \bullet x_2$ which serves as input for NOT-gate to give the outputs as

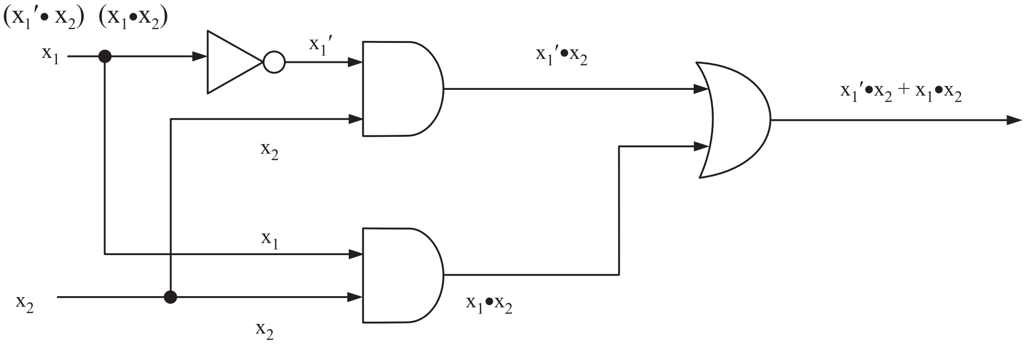
$$(x_1 \bullet x_2)' \qquad \ldots(1)$$

Again the x_1 serving as input for NOT-gate gives x_1' as output. This x_1' along with x_3 serves as input for a OR-gate to give $x_1' + x_3$ as output. This $x_1' + x_3$ along with previous out $(x_1 \bullet x_2)'$ in (1) serve as input for an OR-gate to give $(x_1 \bullet x_2)' + (x_1' + x_3)$ as the final output. Various Boolean expressions and their corresponding Logic circuits are shown below:

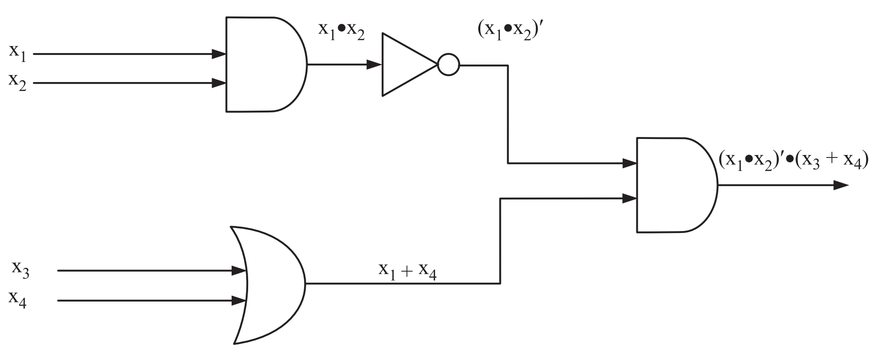
1. $x \bullet y \bullet z + x \bullet y' \bullet z + x \bullet y$ = (output B)
or $(x \wedge y \wedge z) \vee (x \wedge y' \wedge z)\ (x \bullet y) = B$



2. $(x_1' \bullet x_2)\ \ (x_1 \bullet x_2)$



3. $(x_1 \bullet x_2)' \bullet (x_3 + x_4)$

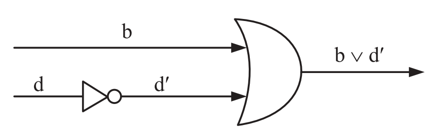


Example 53: Draw the logic circuit to represent the following expression

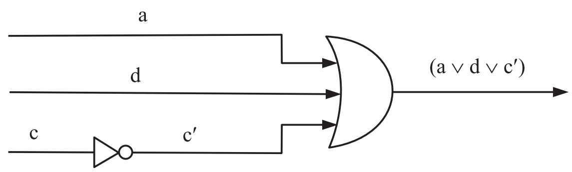
$$f\ (a, b, c, d) = a \wedge [(b \vee d') \vee (c \wedge (a \vee d \vee c'))] \wedge b$$

Solution: This can be done in the following steps by using the rule of precedence.

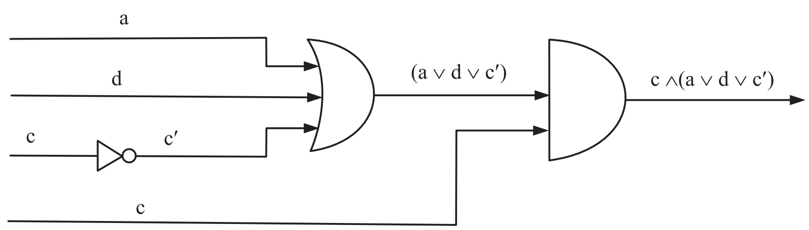
1. $b \vee d'$ can be represented as



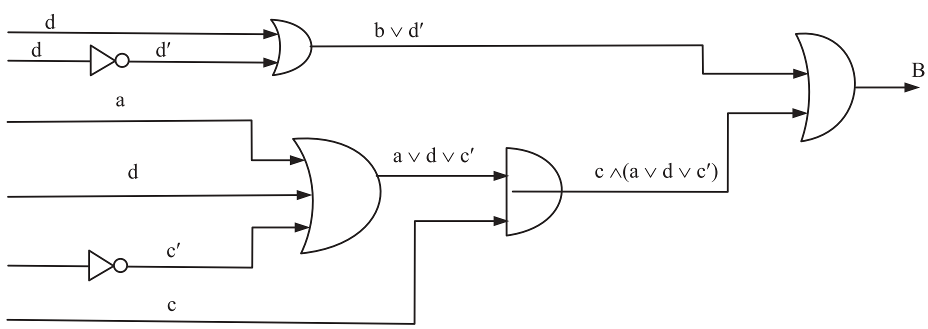
2. $a \vee d \vee c'$ can be represented as



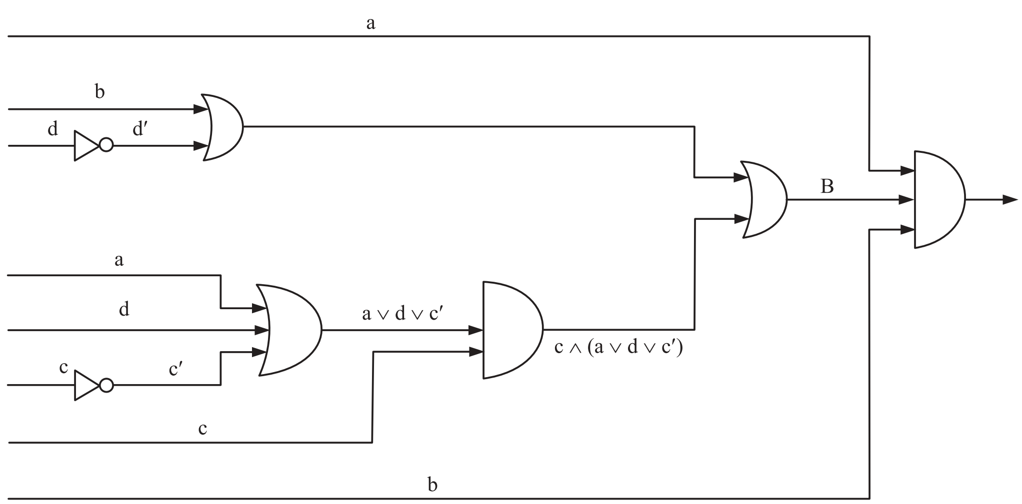
3. $c \wedge (a \vee d \vee c')$ can be represented as



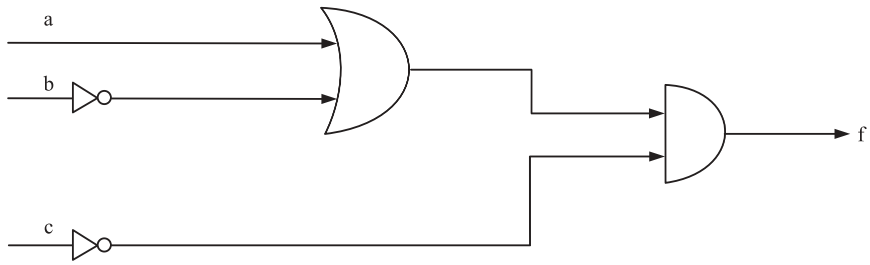
4. Combining steps 1 and 3 the expression $(b \vee d') \vee (c \wedge (a \vee d \vee c')) = B$ can be represented as



5. The expression $a \wedge B \wedge b$ can be represented as



Example 54: Determine the function representing the circuit given below:



Solution: The given function shall be

$$(a + b') \bullet c' \text{ or } (a \vee \bar{b}) \wedge \bar{c}$$

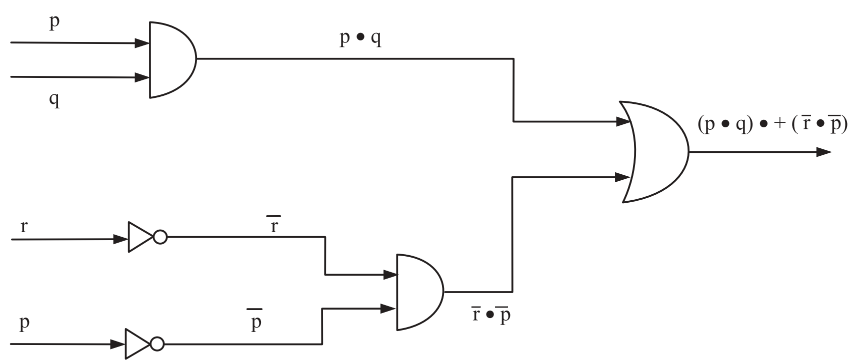
Example 55: For the formula $(p \wedge q) \vee (\neg r \wedge \neg p)$ draw a corresponding circuit diagram by using NOT, AND, OR gates.

Solution: The given expression is

$$(p \wedge q) \vee (\neg r \wedge \neg p) \text{ or } (p \bullet q) + (\bar{r} \bullet \bar{p})$$

which can be represented as follows under the two given conditions:

Using NOT, AND and OR gates



35. ALGORITHM TO FIND THE TRUTH TABLE FOR A LOGIC CIRCUIT L IN WHICH OUTPUT Y IS GIVEN BY A BOOLEAN SUM OF PRODUCTS EXPRESSIONS IN THE INPUTS

Step 1 Write down the sequences for the input A_1, A_2, … and their complements.

Step 2 Find each product appearing in output Y such that a product $x_1 \bullet x_2 \ldots = 1$ in a position in which all the x_1, x_2, … have 1 in the position.

Step 3 Find the sum Y of the product such that $x_1 + x_2 + \ldots = 0$ in a position in which all the x_1, x_2, have 0 in the position.

Example 56: Find out the output Y of the logic circuit represented by $Y = x_1 \bullet x_2 \bullet x_3 + x_1 \bullet x_2' \bullet x_3 + x_1' \bullet x_2$ where $x_1 = 00001111$, $x_2 = 00110011$, $x_3 = 01010101$ are the 8 bit special sequences.

Solution: We have

$$x_1 = 00001111,\ x_2 = 00110011,\ x_3 = 01010101$$

and $$x_1' = 11110000,\ x_2' = 11001100,\ x'_3 = 10101010$$

$$x_1 \bullet x_2 \bullet x_3 = 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1$$

$x_1' \bullet x_2' \bullet x_3$ = 0 0 0 0 0 1 0 0

$x_1' \bullet x_2$ = 0 0 1 1 0 0 0 0

then Y or T(L) = 0 0 1 1 0 1 0 1

36. KARNAUGH MAPS

Karnaugh maps are pictorial devices or graphical methods to determine the prime implicants and minimal form of Boolean expressions involving not more than six variables. Thus it gives us a visual method for simplifying sum of product expressions.

Before explaining the use of Karnaugh maps to simplify sum of product expressions of a Boolean function we shall define a few terms.

We already know that a minterm is a fundamental product involving all the variables and a complete sum of products expression or D.N.F is a sum of distinct minterms.

Two fundamental products P_1 and P_2 are said to **be adjacent if** P_1 and P_2 have the same variables and differ in exactly one literal. It means that there must be an uncomplemented variable in one product and complemented variable in the other product. The sum of two such adjacent products P_1 and P_2 shall be equal to a fundamental product with one less literal as explained in the examples given below:

In each of the following two examples **P_1 and P_2 are adjacent.**

Illustration 1: $P_1 = x \bullet y \bullet z'$, $P_2 = x' \bullet y \bullet z'$

$$P_1 + P_2 = (x + x') \bullet y \bullet z' = 1 \bullet y \bullet z' = yz'$$

Illustration 2: $P_1 = x' \bullet y \bullet z \bullet w$, $P_2 = x' \bullet y \bullet z' \bullet w$

$$P_1 + P_2 = x' \bullet y \bullet w\,(z + z') = x' \bullet y \bullet w$$

Note: If $P_1 = x' \bullet y \bullet z \bullet w$ and $P_2 = x \bullet y \bullet z' \bullet w$ then P_1 and P_2 are not adjacent as these differ in two literals.

Also if $P_1 = x \bullet y \bullet w'$ and $P_2 = x \bullet y \bullet w \bullet z$, then P_1 and P_2 are not adjacent since these have different variables.

Now we shall explain the method to simplify a complete sum of product expression with the help of Karnaugh map for two, three and four variables separately.

37. (A) APPLICATION OF KARNAUGH MAP TO SIMPLIFY A COMPLETE SUM OF PRODUCT EXPRESSION INVOLVING TWO VARIABLES

The map consists of a square divided into four sub-squares as shown in fig (a) below. Let the two variables be x and y. The map is considered like a Venn diagram. Variable x is represented by points in the upper half of the map as shown by shaded portion in fig (b) and x′ is represented by the points in the lower half of the map (shown unshaded in the same fig (b)). Similarly y is represented by the points in the left half of the map as shown by shaded portion in fig (c) and y′ shall be represented by the points in the right half of the map (shown by unshaded portion in the same fig (c)).

In this way the four possible minterms involving two variables i.e. $x \bullet y$, $x \bullet y'$, $x' \bullet y$, $x' \bullet y'$ shall be represented by the points in the four sub-squares in the map as shown below in fig (d).

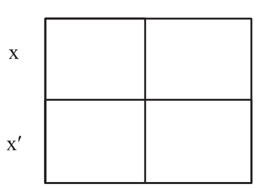


fig (a)

fig (b)

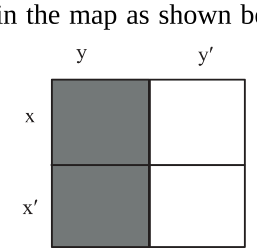


fig (c)

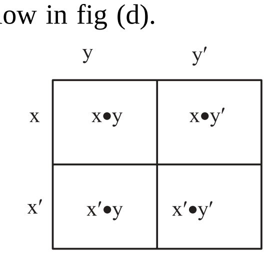


fig (d)

Since a complete sum-of-products expression is a sum of minterms, it can be represented in a Karnaugh map by placing checks in the appropriate squares. We may also place any number say 1 in these squares.

After representing the complete sum-of-product expression on the Karnaugh map, we can determine

(i) **a prime implicant of the given expression** which will be either a pair of adjacent squares (minterms) or an isolated square which is not adjacent to any other square of E(x , y).

(ii) **a minimal sum-of-products** form of E(x, y) which will consist of a minimal number of prime implicants covering all the squares of E(x, y). It means that whenever there are checks (or 1s) in two adjacent squares in the map, the minterms represented by these squares can be combined into a product involving just one of the two variables. For example, $x \bullet y'$ and $x' \bullet y'$ are represented by two adjacent squares which taken together form the right half of the map (which in represented by y'). Therefore $x \bullet y' + x' \bullet y'$ is minimized to y'. If we have check in all the four subsquares, the four minterms shall be represented by the expression 1 involving none of the variables. We draw loops or circle covering the block of sub-squares in the map that represent minterms that can be combined and then find out the corresponding sum of products. Our aim is to identify the largest possible blocks and to include all the checks or 1s with the smallest number of blocks using the largest blocks first and always using the largest possible blocks.

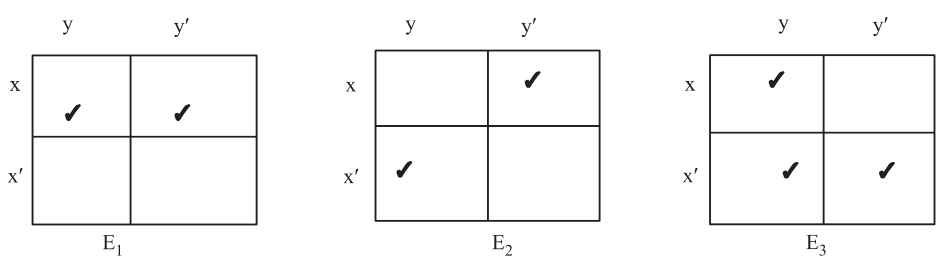
Example 57: Use Karnaugh maps to determine the prime implicants and a minimal sum-of-products form for each of the following complete sum-of-products expressions

(i) $E_1 = x \bullet y + x \bullet y'$

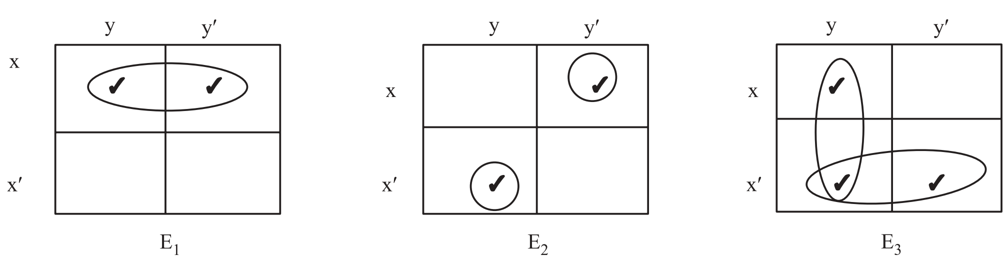
(ii) $E_2 = x \bullet y' + x' \bullet y$

(iii) $E_3 = x \bullet y + x' \bullet y + x' \bullet y'$

Solution: Karnaugh maps showing checks of minterms for the three given expressions are shown below.



The grouping of minterms as shown above, using Karnaugh maps, is represented in the following figures



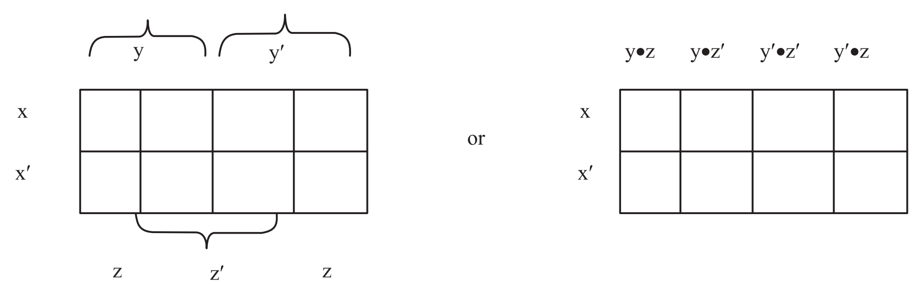
(i) E_1 consists of only one prime implicant comprising of two adjacent squares which are represented by a single variable x. So the only prime implicant of E_1 is x. Therefore the minimal sum-of-product form of $E_1 = x$.

(ii) E_2 consists of two isolated squares representing $x \bullet y'$ and $x' \bullet y$ shown by two loops. Therefore $x \bullet y'$ and $x' \bullet y$ are the two prime implicants of E_2. Thus $E = x \bullet y' + x' \bullet y$ is the minimal sum of product form of E_2.

(iii) The squares containing all the minterms $x \bullet y$, $x' \bullet y$ and $x' \bullet y'$ of E_3 contains two pairs of adjacent squares covered by two loops. The vertical pair of adjacent squares (represented by one loop) represents the variable y and the horizontal pair of adjacent squares (represented by another loop) represents x'. Thus y and x' are the prime implicants of E_3. The minimal sum-of-products form of $E_3 = x' + y$.

[B] APPLICATION OF KARNAUGH MAP TO SIMPLIFY A COMPLETE SUM-OF-PRODUCTS EXPRESSION INVOLVING THREE VARIABLES

The map consists of a rectangle divided into eight squares as shown below. Let the variables be x, y and z. The variable x is represented by the points in the upper half of the map and x' is represented by the points in the lower half of the map. y is represented by the points in the left half of the map and y' by the points in the right half of the map. z is represented by the points in left and right quarters of the map and z' by the points in middle half of the map as shown below:



All the eight minterms involving three variables i.e. $x \bullet y \bullet z$, $x \bullet y \bullet z'$, $x \bullet y' \bullet z'$, $x \bullet y' \bullet z$, $x' \bullet y \bullet z$, $x' \bullet y \bullet z'$, $x' \bullet y' \bullet z'$, $x' \bullet y' \bullet z$ are represented by the points in a square as shown below.

	y z	y • z′	y′ • z′	y′ z
x	$x \bullet y \bullet z$	$x \bullet y \bullet z'$	$x \bullet y' \bullet z'$	$x \bullet y' \bullet z$
x′	$x' \bullet y \bullet z$	$x' \bullet y \bullet z'$	$x' \bullet y' \bullet z'$	$x' \bullet y' \bullet z$

In the Karnaugh map with three variables a basic rectangle denotes either (i) a square or (ii) two adjacent squares or (iii) four squares which form a one-by-four rectangle or two-by-two rectangle.

These basic rectangles corresponds to fundamental products of (i) three or (ii) two or (iii) one literal, respectively. Also the fundamental product represented by a basic rectangle is the product of just those literal that appear in every square of the rectangle.

In order that every pair of adjacent products are geometrically adjacent, the left and right edges of the map are identified by converting the map in the form of a hollow cylinder with left and right edges coinciding.

As in case of two variables we represent a complete sum-of-product expression by placing checks in the appropriate squares of the Karnaugh map. Then a prime implicant of E shall be a maximal basic rectangle (a basic rectangle not contained in any larger basic rectangle of E).

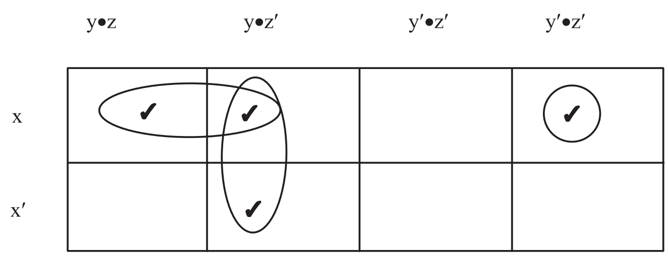
A minimal sum-of-products form for E shall comprise of a minimal number of maximal basic rectangles of E which taken together include all the squares of E.

Example 58 : Find the prime implicants and a minimal sum of products form for each of the following complete sum-of-products expressions.

(i) $E_1 = x•y•z + x•y•z' + x'•y•z' + x•y'•z$

(ii) $E_2 = x•y•z + x'•y•z + x'•y•z' + z•y'•z + x'•y'•z$

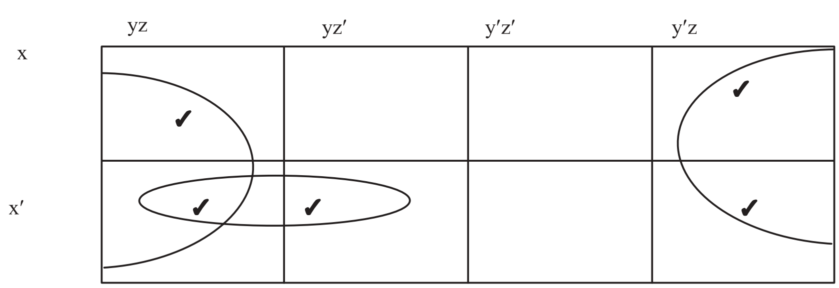
Solution: (i) Checks are placed in the Karnaugh map corresponding to the four minterms in E_1 as shown below



We see the E_1 has three maximal basic rectangles & therefore has three, implicants. There are xy, yz′ and xy′z all of which are needed to cover E. Thus the minimal-sum-of-prodcut form for

$$E = xy + yz' + xy'z.$$

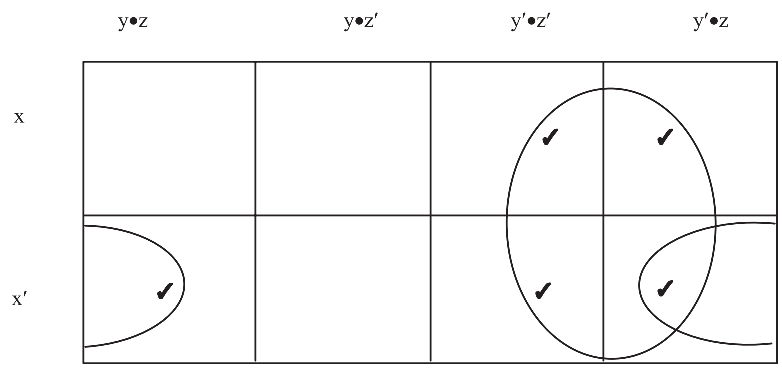
(ii) Checks are placed in the Karnaugh map corresponding to the four minterms of E_2 as shown below



Here E_2 has two prime implicants which have been circled. One is the two adjacent squares representing $x'•y$ ($=x'•y•z + x'•y•z'$) and the other is the two by two square (spanning the first and the last edges) which represents z ($= x•y•z + x'•y•z + x•y'•z + x'•y'•z$). Both these implicants are required to cover E_2. Thus the minimal sum for $E_2 = x•y + z$.

Illustration: The expression $E_3 = x•y'•z + x•y'•z' + x'•y•z + x'•y'•z + x'•y'•z'$ can be represnted as given here. Its minimal form is given by

$$E_3 = y' + x'•z$$



[C] Application of Karnaugh map to simplify sum of product expressions involving four variables:

Karnaugh map for expressions involving four variables is a square divided into sixteen small squares to represent the sixteen possible minterms in four variables as shown below:

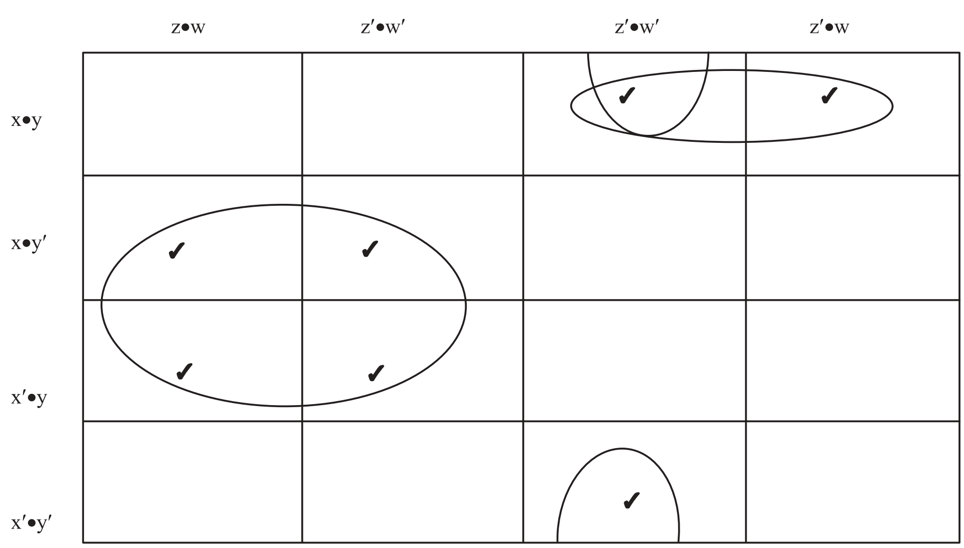
	$y \bullet z$	$y \bullet z'$	$y' \bullet z'$	$y' \bullet z$
$w \bullet x$	$w \bullet x \bullet y \bullet z$	$w \bullet x \bullet y \bullet z'$	$w \bullet x \bullet y' \bullet z'$	$w \bullet x \bullet y' \bullet z$
$w \bullet x'$	$w \bullet x' \bullet y \bullet z$	$w \bullet x' \bullet y \bullet z'$	$w \bullet x' \bullet y' \bullet z'$	$w \bullet x' \bullet y' \bullet z$
$w' \bullet x'$	$w' \bullet x' \bullet y \bullet z$	$w' \bullet x' \bullet y \bullet z'$	$w' \bullet x' \bullet y' \bullet z'$	$w' \bullet x' \bullet y' \bullet z$
$w' \bullet x$	$w' \bullet x \bullet y \bullet z$	$w' \bullet x \bullet y \bullet z'$	$w' \bullet x \bullet y' \bullet z'$	$w' \bullet x \bullet y' \bullet z'$

By definition of adjacent squares, each square is adjacent to four other squares. The simplification of a sum-of-products expression in four variables is achieved by identifying those group of 2, 4, 8, or 16 squares that represent minterms which can be combined.

We give below a sum-of-product expression, its Karnaugh map and it simplified form:

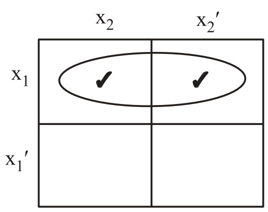
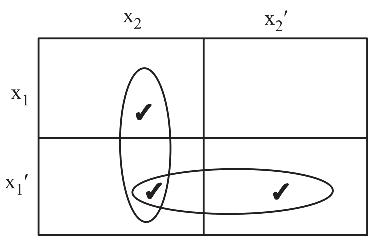
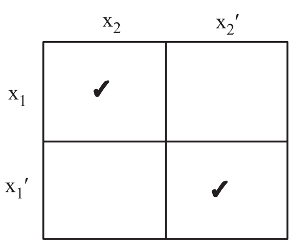
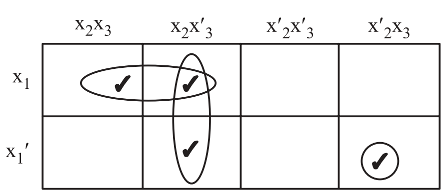
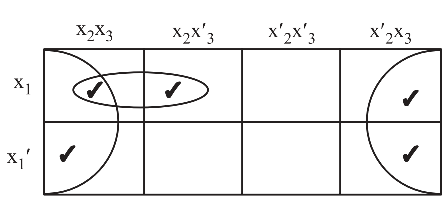
$$E = x \bullet y \bullet z' \bullet w' + x \bullet y \bullet z' \bullet w + x \bullet y' \bullet z \bullet w + x \bullet y' \bullet z \bullet w' + x' \bullet y' \bullet z \bullet w + x' \bullet y' \bullet z \bullet w' + x' \bullet y \bullet z' \bullet w'$$

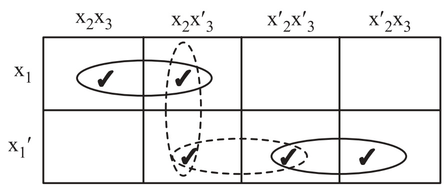
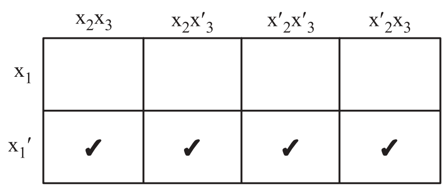
Simplified form is $y' \bullet z + x \bullet y \bullet z' + y \bullet z' \bullet w'$.



38. PRODUCT EXPRESSIONS & K-MAPS

A few sum of product expressions and their K-maps are given below in the first columnof the table. The prime implicants and the minimal sum of product form have also been evaluated from these K maps and given in column 2 and 3 respectively

	Boolean expression and theire K-map	Prime implicants	Equivalent minimal sum of product form
1.	$E_1 = x_1x_2 + x_1x_2'$ 	x_1	x_1
2.	$E_2 = x_1x_2 + x_1' x_2' + x_1' x_2$ 	x_1', x_2'	$x_1' + x_2$
3.	$E_3 = x_1 x_2 + x_1' x_2'$ 	$x_1 x_2, x_1' x_2'$	$x_1 x_2 + x_1' x_2'$
4.	$E_4 = x_1 \cdot x_2 \cdot x_3 + x_1x_2x_3' + x_1' \cdot x_2 \cdot x_3'$ 	$x_1 x_2, x_2 x_3', x_1' x_2' x_3$	$x_1 x_2 + x_2 x_3' + x_1', x_2' x_3$
5.	$E_5 = x_1 x_2 x_3 + x_1 x_2x_3' + x_1x_2' x_3 + x_1' x_2 x_3 + x_1' x_2' x_3$ 	$x_1 x_2, x_3$	$x_1 x_2 + x_3$

	Boolean expression and their K-map	Prime implicants	Equivalent minimal sum of product form
6.	$E_6 = x_1 x_2 x_3 + x_1 x_2 x_3' + x_1' x_2 x_3' + x_1' x_2' x_3' + x_1' x_2' x_3$ 	$x_1x_2, x_2x_3', x_1'x_3'$ and $x_1'x_2'$	$= x_1x_2 + x_1'x_3' + x_1'x_2'$
7.	$x_1'x_2'x_3' + x_1'x_2x_3' + x_1'x_2'x_3 + x_1'x_2x_3$ 	x_1'	x_1'

Note: The minimal sum of products form obtained in the above table from K-map can also be determined algebraically by using Boolean Algebra laws.

EXERCISES

1. The two operations + and • are defined on a non empty set B = {p, q, r, s} as follows:

+	p	q	r	s
p	p	q	q	p
q	q	q	q	q
r	q	q	r	r
s	p	q	r	s

•	p	q	r	s
p	p	p	s	s
q	p	q	r	s
r	s	r	r	s
s	s	s	s	s

Prove that (B, +, •) is a Boolean algebra.

[**Hint:** s is the identity for + and q is the identity of •]

2. Let S be a set of positive divisors of 30 and the operations $\vee$ and $\wedge$ are defined as

$x \vee y = u$, the L.C.M. of a, b

and $x \wedge y = v$, the H.C.F. of a, b $\forall$ x, y, u, v $\in$ S

Prove that (S, $\vee$, $\wedge$) is a Boolean algebra.

[**Hint:** S = {1, 2, 3, 5, 6, 10, 15, 30}, 1 is the identity for $\wedge$ and 30 is the identity for $\vee$ complement of x shall be y such that $x \vee y = 30$ and $x \wedge y = 1$]

3. If (B, +, •, /) is a Boolean algebra and a, b $\in$ B then prove that

(i) $a' + a \bullet b = a' + b$

(ii) $(a + b)' + (a + b')' = a'$

[**Hint:** (ii) $(a + b)' + (a + b')'] = [(a + b)\bullet(a + b')]'$ (by Demorgan's law)
$= (a + b\bullet b')'$ (by distributive law)
$= (a + 0)' = a']$

4. If (B, +, •, /) is a Boolean algebra, prove that

(i) $a\bullet b + a\bullet b' + a'\bullet b + a'\bullet b' = 1 \ \forall \ a, b \in B$.

(ii) $a\bullet b + a'\bullet b' = (a' + b) \bullet (a + b') \ \forall \ a, b \in B$.

5. If $(a + x) = b + x$ and $a + x' = b + x'$, prove that $a = b$ where $a, b, x \in B$ and (B, +, •, /) is a Boolean algebra.

6. Find C.N.F. of $X(x_1, x_2, x_3) = ((x_1 \wedge x'_2) \vee (x_1' \wedge x'_3))'$

[**Ans:** $(x_1' \vee x_2 \vee x_3) \wedge (x_1' \vee x_2 \vee x'_3) \wedge (x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee x'_2 \vee x_3))$

7. Obtain the disjunctive normal form (join of minterms) of the following Boolean expression.

$(x_1 \wedge x_2^c) \vee (x_2 \wedge x_3^c) \vee (x_3 \wedge x_1^c)$ where subscript c represents complement.

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[**Hint:** $x_1 \wedge x'_2 = (x_1 \wedge x_2') \wedge I = (x_1 \wedge x'_2) \wedge (x_3 \vee x'_3)$
$= (x_1 \wedge x'_2 \wedge x_3) \vee (x_1 \wedge x'_2 \wedge x'_3)$ …(1)

Similarly $x_2 \wedge x_3' = (x_2 \wedge x_3') \wedge I = (x_2 \wedge x'_3) \wedge (x_1 \vee x'_1)$
$= (x_2 \wedge x_3' \wedge x_1) \vee (x_2 \wedge x'_3 \wedge x_1')$
$= (x_1 \wedge x_2 \wedge x_3') (x_1' \wedge x_2 \wedge x'_3)$ …(2)

and $x_3 \wedge x'_1 = (x_3 \wedge x'_1) \wedge I = (x_3 \wedge x'_1) \wedge (x_2 \vee x'_2)$
$= (x_3 \wedge x'_1 \wedge x_2) \vee (x_3 \wedge x'_1 \wedge x_2')$
$= (x_1' \wedge x_2 \wedge x_3) \vee (x'_1 \wedge x'_2 \wedge x_3)$ …(3)

The required result is (1) ∨ (2) ∨ (3).

8. Express the following functions in disjunctive normal form.

(a) $X_1(x, y, z) = [x + y' + (y + z)']' + yz$

(b) $X_2(x, y, z) = [(x + y) (z'y')']$

[**Ans:** (a) $x'yz + x'yz' + xyz$ (b) $xyz + xy'z + xyz' + x'yz + x'yz'$]

9. Express the following expression in DNF in the smallest possible number of variables $(a + b)(a + b')(a' + c)$. Also find DNF in the variables a, b, c.

[**Hint:** $(a + b)(a + b')(a' + c) = (aa' + ab' + ba + bb')(a' + c)$
$= (a + ab' + ab)(a' + c) = aa' + ac + ab'c' + abc$
$= ac + abc = ac(1 + b) = ac\bullet 1 = ac$.

which is DNF in 2 variables. Again $a\bullet c = ac(b + b') = acb + acb' = abc + ab'c$.]

10. Simplify $(x + y)(x + z)(x'y')'$

[**Hint:** $(x + y)(x + z)(x + y) = (x + y)(x + z)$
$= x\bullet x + x\bullet z + y\bullet x + y\bullet z = x + xy + yz = x + yz$]

11. Express the following of Boolean expressions in C.N.F

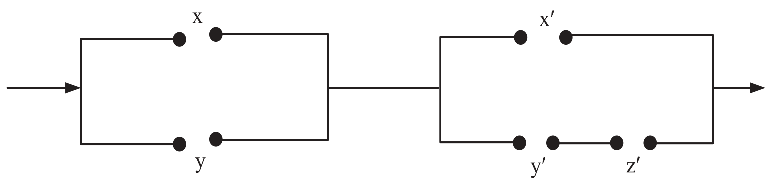
(i) $x' + yz$ (ii) $xy + x'y'$

[**Ans:** (i) $(x' + y + z)(x' + y + z')(x + y + z)(x' + y' + z)$
(ii) $(x' + y)(x + y')$]

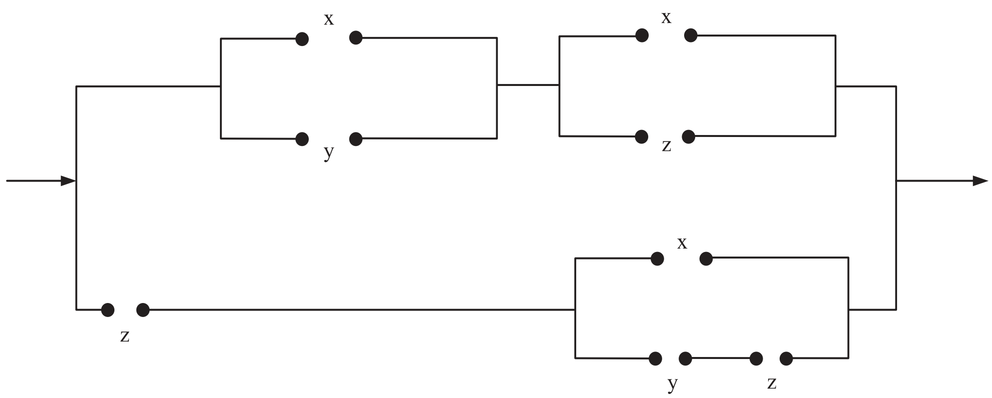
12. Draw the swithing circuit of the following Boolean expression

$X(x, y, z) = (x + y) \bullet (x' + y'\bullet z')$

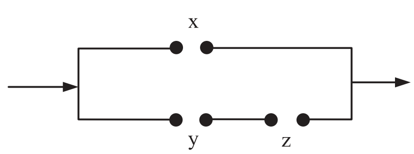
[**Ans:**



13. Find the Boolean expression of the following switching circuit. Find its equivalent simplified circuit



[**Ans**: (x + y)•(x + z) + z•(x + y + z) Simplified expression is x + y•z



14. Find the prime implicants and a minimal sum-of-products form for each of the following complete sum-of-product expressions

(i) E_1 = xyz + xyz′ + x′yz′ + x′y′z

(ii) E_2 = xyz′ + xy′z′ + x′yz + x′y′z′

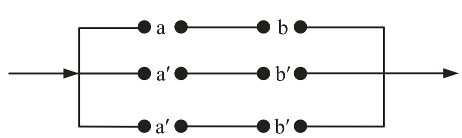
(iii) E_3 = xyz + xyz′ + x′yz′ + x′y′z′ + x′y′z

[**Ans:** (i) E_1 = xy + yz′ + x′•y′z

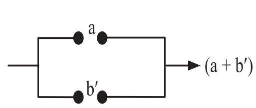
(ii) E_2 = xz′ + y′z′ x′y′z′

(iii) E_3 = xy + x′z′ + x′y′]

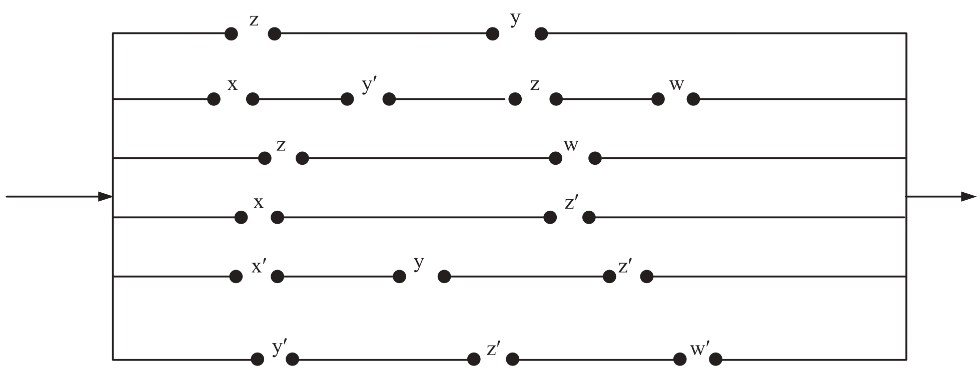
15. Simplify the switching circuit given below and show that the two circuits are equivalent by using truth table.



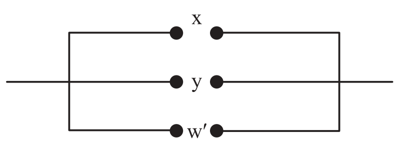
[**Ans:**



16. Give the simplified form of the following circuit

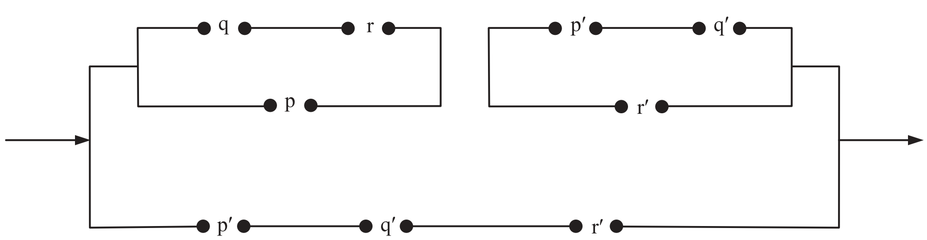


[**Ans:**:



$x + y + w'$

17. Draw the following network into simplified form: **[C.C.S.U., M.Sc. (Maths) 2004]**



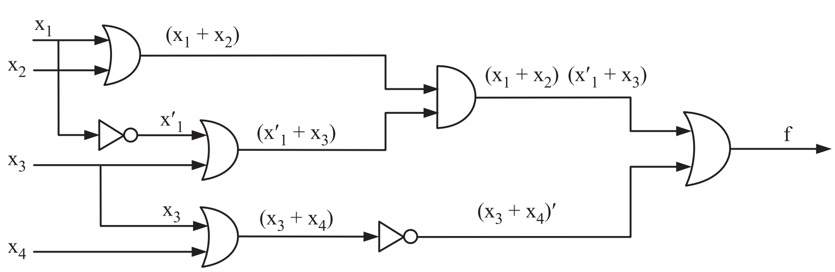
[**Hint:** The Boolean expression corresponding to the given network is

$(q \bullet r + p) \bullet (p' q' + r) + p' q' r'$

18. Construct a circuit using gates to realize the Boolean expression:

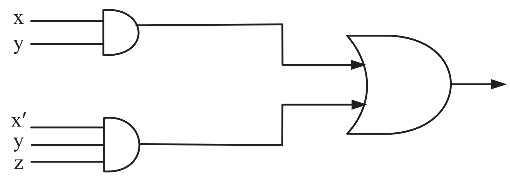
$f = (x_1 + x_2) \bullet (x_1' + x_3) + (x_3 + x_4)'$ **[C.C.S.U., M.Sc. (Maths) 2004]**

[**Hint:**



19. Write the Boolean function corresponding to the following network

[C.C.S.U., M.Sc. (Maths) 2005]



[**Hint:** $x \bullet y + x' \bullet y \bullet z$]

20. Find the duals of (i) $x' \bullet 1 + (y' + z)$ and (ii) $a \bullet (b + 0)$

[**Ans:** (i) $(x' + 0) \bullet (y' \bullet z)$ (ii) $a + (b \bullet 1)$

21. Construct an identity from the absorption law $a \bullet (a + b) = a$ by taking duals

[**Ans:** $a + a \bullet b = a$]

22. If the Boolean operator $\oplus$ called XOR operator is defined as $1 \oplus 1 = 0$, $1 \oplus 0 = 1$, $0 \oplus 1 = 1$ and $0 \oplus 0 = 0$, then prove that

$$x \oplus y = (x + y)\ (x\,y)'$$

$$x \oplus y = (x \bullet y') + (x' \bullet y)$$

[**Hint:**

x	y	$x \oplus y$	x + y	x y	(x y)′	(x + y) • (x y)′	x y′	x′ y	xy′ + x′y
0	0	0	0	0	1	0	0	0	0
0	1	1	1	0	1	1	0	1	1
1	0	1	1	0	1	1	1	0	1
1	1	0	1	1	0	0	0	0	0

23. Apply rules of Boolean algebra to prove that

(i) $(a \vee b) \wedge (a' \vee b) \equiv y$

(ii) $[a \wedge c) \vee (b' \vee c)'] \vee [(b \wedge c) \vee (a \wedge c')] \equiv a + b$

24. If B = {0, 1}, compute truth table for the Boolean function $f : B_3 \to B$ determined by the Boolean expression

(i) $p(a, b, c) = (a \wedge b') \vee (b \wedge (a' \vee b))$

(ii) $p(a, b, c) = a \wedge (b \vee c')$

Also construct the logic diagram implementing these functions.

25. Show that in a Boolean algebra, for any $x, y, z \in B$

$$\left((x \vee z)' \wedge (y' \vee z)\right)' = (x' \vee y) \wedge z'$$

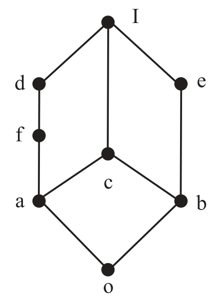
[**Hint:** LHS $= (x \vee z)' \vee (y' \vee z')$ (by De Morgan's law)

$= (x' \wedge z') \vee (y \wedge z')$ (by De Morgan's law)

$= (x' \vee y) \wedge z'$ (by distribution law)

= RHS.

26. Is the lattice whose Hasse diagram has been given below a Boolean algebra



[**Hint:** d and f both are the complements of b. It mean complement of b is not unique. Hence it is not a Boolean algebra.

MULTIPLE CHOICE QUESTIONS (MCQs)

1. The minimum number of elements in a Boolean algebra is
(a) 1 (b) 2 (c) 3 (d) 4

2. Idempotent law in Boolean algebra is
(a) $(a')' = a$ (b) $a + a \bullet b = a$ (c) $a + a = a$ (d) $a + 1 = a$

3. In Boolean algebra, if $a, b \in B$, then absorption law is
(a) $a + (a \bullet b) = a$ (b) $a \bullet a = a$ (c) $a + a = a$ (d) none of these

4. In Boolean algebra $a \bullet a = a$ is known as
(a) De-Morgan's law (b) Absorption law (c) Idempotent law (d) none of these

5. In Boolean algebra $a \bullet (a + b) = a$ is known as
(a) Idempotent law (b) Absorption law (c) De-Morgan's law (d) none of these

6. In Boolean algebra $a + (b \bullet c) = (a + b) \bullet (a + c)$ follows from
(a) Distributive law (b) Associative law (c) Idempotent law (d) none of these

7. For any two elements a and b in Boolean algebra $(a + b)' = a' \bullet b'$ and $(a \bullet b)' = a' + b'$ are known
(a) Idempotent law (b) Absorption law (c) De-Morgan's law (d) none of these

8. In Boolean algebra, the dual of $a \bullet 0 = 0$ is
(a) $a + 1 = 1$ (b) $a \bullet 0 = 1$ (c) $0 \bullet a = 0$ (d) none of these

9. In Boolean algebra, the dual of $a \bullet (b + c) = (a \bullet b) + (a \bullet c)$ is
(a) $a + (b \bullet c) = (a + b) \bullet (a + c)$ (b) $(b + c) \bullet a = (a \bullet c) + (a \bullet b)$
(c) $a \bullet b + a \bullet c = (a \bullet b) + (a \bullet c)$ (d) none of these

10. In Boolean algebra, the dual of $a \bullet a' = 0$ is
(a) $a + a' = 1$ (b) $a' \bullet a = 0$ (c) $a \bullet a' = 1$ (d) none of these

11. In Boolean algebra, which of the following statements is true for $x, y \in B$
(a) $(x \bullet y)' = x' - y'$ (b) $(x \bullet y)' = x' + y'$ (c) $(x \bullet y)' = x' \bullet y'$ (d) none of these

12. In Boolean algebra, if $a, b \in B$, then $a + a'b$ is equal to
(a) $a + b$ (b) $a' + b'$ (c) $a + ab'$ (d) none of these

13. In Boolean algebra, if $a, b \in B$, then value of $a' + a \bullet b$ is equal to
(a) $a + b'$ (b) $a' + b$ (c) $a' + b'$ (d) none of these

14. In Boolean algebra, if $a, b \in B$, then value of $a \bullet (a \bullet b)$ is
(a) $a \bullet a$ (b) $b + a$ (c) $a \bullet b$ (d) none of these

15. Which of the following statement is true in Boolean algebta, where $a, b \in B$
(a) $(x + y)' = x' \bullet y'$ (b) $(x + y)' = x' + y'$ (c) $(x + y)' = x \bullet y$ (d) none of these

16. In Boolean algbera if $a \in B$, then
(a) $(a')' = a'$ (b) $(a')' = a \bullet a$ (c) $(a')' = a$ (d) none of these

17. Which of the following statements is false in Boolean algebra, where $a, b \in B$
(a) $a \bullet (a + b) = b$ (b) $(a')' = a$ (c) $a + a = a$ (d) $a \bullet a = a$

18. Which of the following statements are true in Boolean algebra if $a \in B$
(a) $a + 1 = a$ (b) $a + 1 = 1$ (c) $a \bullet 0 = a$ (d) $a \bullet 0 = 1$

19. Which of the following statement is true in Boolean algebra if $x \in B$
(a) $x + x' = 1$ (b) $x + 0 = 0$ (c) $x \bullet 1 = 1$ (d) $x \bullet x' = 1$

20. A Boolean algebra can not have
(a) 2 elements (b) 3 elements (c) 4 elements (d) 5 elements

21. Simplified form of the switching function $F(x, y) = x + x \bullet y$ is
(a) $x \bullet y$ (b) x (c) y (d) $x + y$

22. Simplified form of the switching function $F(x, y, z) = x \bullet y + y \bullet z + y \bullet z'$ is
(a) y (b) x (c) $x \bullet y$ (d) none of these

23. The Boolean expression $x \bullet y' + x \bullet y + x \bullet y' + x' y$ is equivalent to
(a) 0 (b) 1 (c) x y (d) none of these

24. l.u.b. of the elements a and b of a Boolean algebra B is
(a) $a + b$ (b) 1 (c) 0 (d) $a \bullet b$

25. g.l.b. {a, b} of a Boolean algebra B is
(a) $a + b$ (b) $a \bullet b$ (c) 1 (d) 0

26. Complete D.N.F. of a Boolean function in two variables p and q is
(a) $p \bullet q + p' \bullet q + p \bullet q' + p' \bullet q'$ (b) $p \bullet q' + p' q$
(c) $p \bullet q$ (d) $p \bullet p + q \bullet q + p' \bullet q + p \bullet q'$

27. The complement of Boolean function $F(x, y) = x' y + x y' + x' y'$ is
(a) $x \bullet y$ (b) $x + y$ (c) $(x \bullet y)'$ (d) none of these

ANSWERS

1. (b)	**2.** (c)	**3.** (a)	**4.** (c)	**5.** (b)	**6.** (a)	**7.** (c)
8. (a)	**9.** (a)	**10.** (a)	**11.** (b)	**12.** (c)	**13.** (b)	**14.** (c)
15. (a)	**16.** (c)	**17.** (a)	**18.** (b)	**19.** (a)	**20.** (b)	**21.** (b)
22. (a)	**23.** (b)	**24.** (a)	**25.** (b)	**26.** (a)	**27.** (a)	