2 Particle Kinematics

Kinematics studies the *geometry of motion* without reference to the causes of motion. It is rooted in vector algebra and calculus, and to many "it looks and feels like math." Kinematics is essential for the application of Newton's second law, $\vec{F} = m\vec{a}$, since it allows us to describe the $\vec{a}$ in $\vec{F} = m\vec{a}$. Unfortunately, just writing $\vec{F} = m\vec{a}$ doesn't tell us how something moves; it only describes the relationship between force and acceleration. In general, we also want to know a particle's *motion*, where by *motion* we mean "all the positions occupied by an object over time." It is kinematics that allows us to translate $\vec{a}$ into the *motion* of a point by using calculus. This is why in this chapter we study the concepts of position, velocity, and acceleration, and how these quantities relate to one another. In addition, we will learn how to write position, velocity, and acceleration in the various component systems most commonly used in dynamics.

2.1 Position, Velocity, Acceleration, and Cartesian Coordinates

Motion tracking along a racetrack

A motion tracking system like those used by TV networks during sporting events for the motion analysis of a football or a hockey puck can be used to analyze the motion of race cars along tracks such as that in Fig. 2.1. Two of the

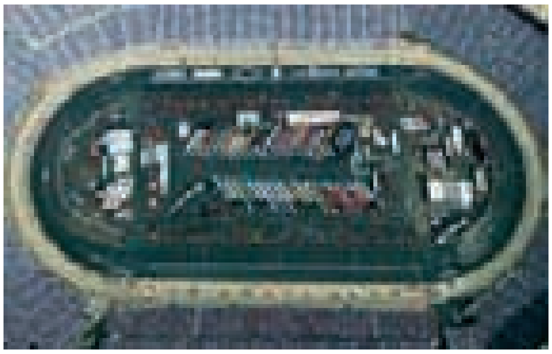

Figure 2.1. A car racetrack.

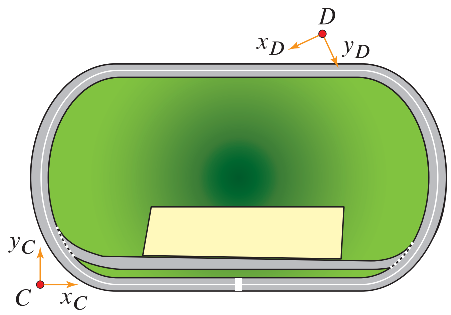

Figure 2.2
Tracking cameras C and D. The arrows at a camera indicate the directions of the Cartesian component system used by that camera.

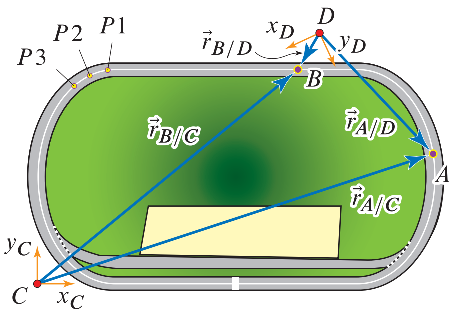

Figure 2.3
Position *vectors* of a car at two distinct times. Vectors $\vec{r}_{A/C}$ and $\vec{r}_{B/C}$ describe the position of a car at times $t_1 = 42.53$ s and $t_2 = 45.76$ s, respectively, as seen by camera C (see Table 2.1). Similarly, $\vec{r}_{A/D}$ and $\vec{r}_{B/D}$ are the positions of the *same* car and times relative to camera D.

cameras, C and D, that are part of the tracking system are shown in Fig. 2.2. The cameras can follow a car and measure the distance to it. The tracking information is then fed to a motion analyzer, which records it as in Table 2.1, i.e., as coordinates taken at regular time intervals. These coordinates are relative to the axes indicated in Fig. 2.2. How can we use the given data to learn how fast a car is moving and what acceleration the car is subject to?

Table 2.1. Cartesian coordinates of a car during a test run (only camera C and D data is reported). The system is able to measure time to within 0.01 s and distance to 0.1 ft.

Position	Time (s)	(x_C, y_C) (ft)	(x_D, y_D) (ft)
A	42.53	(1098.4, 360.6)	(-143.9, 437.2)
B	45.76	(723.2, 594.4)	(97.3, 66.7)
$P1$	48.87	(195.6, 593.0)	(576.1, -155.0)
$P2$	49.26	(145.6, 577.7)	(627.9, -162.3)
$P3$	49.66	(102.5, 548.8)	(679.2, -154.3)

Let's begin by observing that the locations of a car can be represented by arrows going from the camera that has recorded them to the locations in question (see Fig. 2.3). This suggests that we can use *vectors* to describe the positions of points. As can be seen in Fig. 2.3, the magnitude and direction of these vectors depend on the reference point (i.e., camera) that defines them.

The idea of velocity stems from computing the ratio between a change in position and the time interval spanned by the position change. Therefore we could measure a car's velocity by computing the difference between the coordinate pairs from two consecutive rows in Table 2.1 and dividing by the corresponding time difference. This operation raises various questions: If we take differences between two positions and then divide by the corresponding time interval, are we calculating the velocity at a *specific location* or are we calculating the velocity over some *range of positions?* Can we ever talk about "the velocity of a car at some specific location"? If position is described by a vector, is velocity also described by a vector? Should we expect the two velocity vectors measured by two different cameras to be different since position vectors relative to these cameras are different?

The questions raised are at the core of *kinematics* and tell us that to deal with any engineering application concerning the description of motion, we need rigorous definitions of the concepts of position, velocity, and acceleration. In the remainder of this section we will see that vectors play a fundamental role in how we define position, velocity, and acceleration and how we can correctly interpret information provided from multiple observation reference points. As we formally define position, velocity, and acceleration, we will come back to the car tracking problem to provide concrete applications to our definitions.

A notation for time derivatives

In studying kinematics, we will be writing derivatives with respect to time so often that it is convenient to have a shorthand notation. If $f(t)$ is a function of time, we write a dot over it to mean $df(t)/dt$. Furthermore, the *number* of

$\vec{r}(t_j) - \vec{r}(t_i)$. In general, the length of $\Delta\vec{r}(t_i, t_j)$ *does not* measure the distance traveled by P between t_i and t_j (highlighted in yellow in Fig. 2.5). This is so because the distance traveled between t_i and t_j depends on the geometry of the path of P whereas $\Delta\vec{r}(t_i, t_j)$ depends only on $\vec{r}(t_i)$ to $\vec{r}(t_j)$ without reference to *how* P moved from one position to the other!

Average velocity vector. We define the *average velocity vector* of P over the time interval (t_i, t_j) as

$$\vec{v}_{\text{avg}}(t_i, t_j) = \underbrace{\frac{1}{t_j - t_i}}_{\text{scalar}} \underbrace{\left[\vec{r}(t_j) - \vec{r}(t_i)\right]}_{\text{vector}} = \frac{\Delta\vec{r}(t_i, t_j)}{t_j - t_i}. \tag{2.6}$$

Let's observe that the vectors $\vec{v}_{\text{avg}}(t_i, t_j)$ and $\Delta\vec{r}(t_i, t_j)$ have the same direction since the term $1/(t_j - t_i)$ in Eq. (2.6) is a positive scalar.

Concept Alert

Velocity vector. The velocity *vector* is a *vector* because it is the time rate of change of the position *vector*.

Velocity vector. Considering the average velocity over a time interval $(t, t + \Delta t)$, we define the *velocity vector* at time t as

$$\vec{v}(t) = \lim_{\Delta t \to 0} \vec{v}_{\text{avg}}(t, t + \Delta t) = \lim_{\Delta t \to 0} \frac{\Delta\vec{r}(t, t + \Delta t)}{(t + \Delta t) - t}. \tag{2.7}$$

Calculus tells us that the second limit in Eq. (2.7) is the time derivative of $\vec{r}(t)$, so that the velocity vector is normally written as

$$\boxed{\vec{v}(t) = \frac{d\vec{r}(t)}{dt} = \dot{\vec{r}}(t),} \tag{2.8}$$

that is, *The velocity vector is the time rate of change of the position vector*.

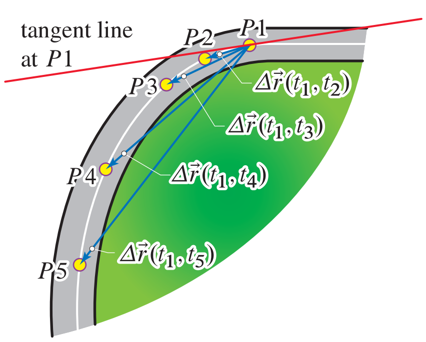

Figure 2.6
Displacement vectors $\Delta\vec{r}(t_i, t_j)$ between time position $P1$ at time t_1 and subsequent positions and times Pi at t_i, $i = 2, 3, 4, 5$.

Speed. The *speed* of a point is defined as the *magnitude of its velocity*:

$$\boxed{v(t) = \left|\vec{v}(t)\right|.} \tag{2.9}$$

Therefore, by definition, the *speed is a scalar quantity that is never negative*.

The velocity vector is *always* tangent to the path. The velocity vector has an important property: *The velocity vector at a point along the trajectory is tangent to the trajectory at that point!* To see this, recall that after Eq. (2.6) we remarked that the average velocity between two time instants, say, t and $t + \Delta t$, has the same direction as the displacement vector $\Delta\vec{r}(t, t + \Delta t)$. Figure 2.6 illustrates that as $\Delta t \to 0$, the vector $\Delta\vec{r}(t, t + \Delta t)$ becomes *tangent* to the trajectory, thus causing the velocity vector to become tangent to the trajectory at the position $\vec{r}(t)$.

Concept Alert

Direction of velocity vectors. One of the most important concepts in kinematics is that the velocity of a particle is always tangent to the particle's path.

Additional properties of the velocity vector. We can now go back and answer some of the questions raised by the car tracking problem. Specifically, if, using Table 2.1, we compute the ratio between a position change and the corresponding time interval, then (1) we now know that what we are actually computing is the car's *average velocity* and (2) the average velocity cannot be said to pertain to any particular position within the time interval considered. We can also answer the question about whether or not the data collected from two different cameras yields different velocity vectors. Observe that the vector

EXAMPLE 2.1 *How Do You Get to Carnegie Hall? . . . Practice!*

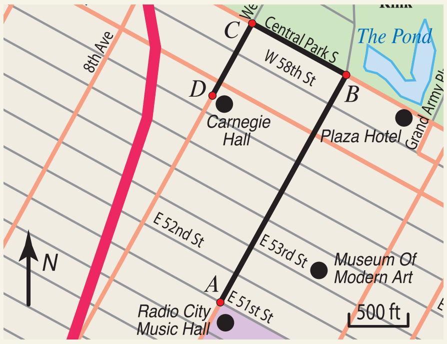

Figure 1
Cab route.

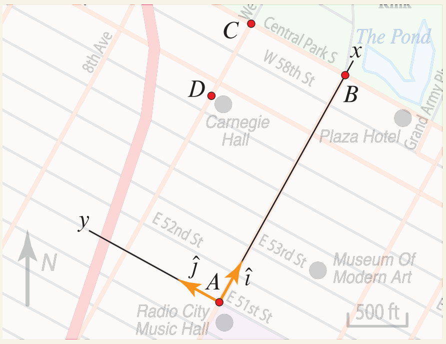

Figure 2
Cartesian coordinate system with origin at the pickup point A. The city grid is such that the line through B and C is parallel to the y axis and the line through C and D is parallel to the x axis.

Table 1
Coordinates of the points defining the cab's route.

Point	x (ft)	y (ft)
A	0	0
B	2200	0
C	2200	906
D	1500	906

A cab picks up a passenger outside Radio City Music Hall on the corner of the Avenue of the Americas and E 51st St. (point A) and drops her off in front of Carnegie Hall on 7th Ave. (point D) after 5 min, following the route shown. Find the cab's displacement, average velocity, distance traveled, and average speed in going from A to D. Note that the distance from A to B is 2200 ft, from B to C is 906 ft, and from C to D is 700 ft.

SOLUTION

Road Map To solve the problem, we need to set up a coordinate system and identify the coordinates of points A, B, C, and D that define the cab's path. Then the problem's questions can be answered by applying the definitions of displacement, average velocity, distanced traveled, and speed.

Computation Referring to Fig. 2, we select a Cartesian coordinate system with origin at A and aligned with the city grid. Using the given information, the coordinates of A, B, C, and D are given in Table 1. Since the displacement from A to D is the difference between the position vectors of points A and D, we have

$$\boxed{\Delta\vec{r}(t_A, t_D) = \vec{r}(t_D) - \vec{r}(t_A) = (1500\,\hat{\imath} + 906\,\hat{\jmath})\text{ ft},} \tag{1}$$

where t_A and t_D are times at which the cab is at A and D, respectively. Applying the definition of average velocity in Eq. (2.6), for the time interval (t_A, t_D) we have

$$\boxed{\vec{v}_{\text{avg}}(t_A, t_D) = \frac{\Delta\vec{r}(t_A, t_D)}{t_D - t_A} = \frac{\vec{r}(t_D) - \vec{r}(t_A)}{t_D - t_A} = (5\,\hat{\imath} + 3.02\,\hat{\jmath})\text{ ft/s},} \tag{2}$$

where $t_D - t_A = 5\text{ min} = 300\text{ s}$. Next, the distance traveled by the cab, which we will denote by d, is given by the sum of the lengths of the segments $\overline{AB}$, $\overline{BC}$, and $\overline{CD}$, i.e,

$$\boxed{d = (x_B - x_A) + (y_C - y_B) + (x_C - x_D) = 3810\text{ ft}.} \tag{3}$$

Since the speed is the magnitude of the velocity, the *average speed* must be computed as the *average of the magnitude of the velocity*, i.e.,

$$v_{\text{avg}} = \frac{1}{t_D - t_A}\int_{t_A}^{t_D} |\vec{v}|\,dt = \frac{1}{t_D - t_A}\left(\int_{t_A}^{t_B} v_x\,dt + \int_{t_B}^{t_C} v_y\,dt - \int_{t_C}^{t_D} v_x\,dt\right), \tag{4}$$

since v is equal to v_x, v_y, and $-v_x$ during the time intervals (t_A, t_B), (t_B, t_C), and (t_C, t_D), respectively. Now notice that the last three integrals in Eq. (4) measure the distance traveled by the cab in each of the corresponding time intervals. Considering the integral, say, over (t_A, t_B), we can write

$$\int_{t_A}^{t_B} v_x\,dt = \int_{t_A}^{t_B} \frac{dx}{dt}\,dt = \int_{x_A}^{x_B} dx = x_B - x_A. \tag{5}$$

Proceeding similarly for the other two integrals, we find the average speed to be

$$\boxed{v_{\text{avg}} = \frac{(x_B - x_A) + (y_C - y_B) - (x_D - x_C)}{t_D - t_A} = \frac{d}{t_D - t_A} = 12.7\text{ ft/s}.} \tag{6}$$

Discussion & Verification The results obtained are dimensionally correct. Since 12.7 ft/s corresponds to 8.66 mph, we can consider the result acceptable since such a speed is typical of city traffic such as can be found in midtown Manhattan.

A Closer Look Observing that $|\Delta\vec{r}(t_A, t_D)| = 1750$ ft, and that $|\vec{v}_{\text{avg}}(t_A, t_D)| = 5.84$ ft/s, we see this example reinforces the idea that we should never confuse distance traveled for displacement or average velocity for average speed.

EXAMPLE 2.2 Trajectory, Velocity, and Acceleration

Two stationary ships A and B track an object P launched from A and flying low and parallel to the water. Relative to the Cartesian frames A and B in Fig. 1 and for the first few seconds of flight, the recorded motion of P is

$$\vec{r}_{P/A}(t) = 2.35t^3\,\hat{\imath}_A \text{ m}, \tag{1}$$

$$\vec{r}_{P/B}(t) = \left[\left(225 + 2.13t^3\right)\hat{\imath}_B + \left(225 + 0.993t^3\right)\hat{\jmath}_B\right]\text{m}, \tag{2}$$

where t is in seconds.

(a) Determine the path of P as viewed by frame A and frame B.

(b) Using *both* $\vec{r}_{P/A}$ and $\vec{r}_{P/B}$, determine the velocity and the speed of P.

(c) Find the acceleration and its orientation in relation to the path in both frames A and B.

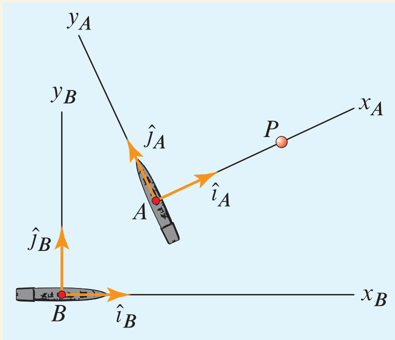

Figure 1
Two Cartesian frames A and B.

SOLUTION

Part (a): Path of P

Road Map We have the motion of P as a function of time. Since the path of P is the line traced by P in *space*, we can find the path by eliminating time from the motion.

Computation Starting with the motion in component form for frame B, we have

$$x_{P/B}(t) = \left(225 + 2.13t^3\right)\text{m} \quad \text{and} \quad y_{P/B}(t) = \left(225 + 0.993t^3\right)\text{m}. \tag{3}$$

To eliminate the time variable, we solve the first of Eqs. (3) for t^3 and then substitute the result in the second of Eqs. (3). This yields

$$\boxed{y_{P/B} = \left[225 + \frac{0.993}{2.13}\left(x_{P/B} - 225\right)\right]\text{m} = \left(120 + 0.466x_{P/B}\right)\text{m}.} \tag{4}$$

Observe that in frame A, $y_{P/A} = 0$ at all times so that no calculation is needed to eliminate time from the y component of the motion of P. Therefore, the path of P in frame A is described by $y_{P/A} = 0$, which is the equation of the x_A axis.

Helpful Information

Trajectory and time. The trajectory (or path) is what the motion looks like once time is removed from the motion's description.

Discussion & Verification The path in frame B is given in Eq. (4), which is the equation of a straight line. This agrees with the calculation of the path in frame A, in which the path lies on the x_A axis, which is also a straight line. The trajectory as seen by the two frames is shown in Fig. 2. Note that Fig. 2 also indicates the location of P at $t = 0$ and the direction of motion.

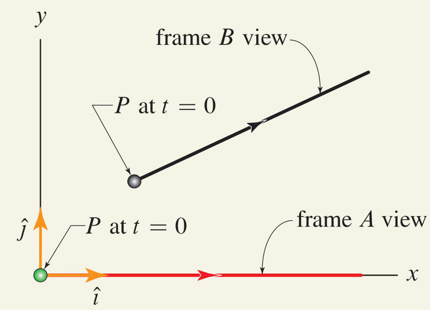

Figure 2
Path of P as seen by frames A and B.

Part (b): Velocity and Speed of P

Road Map We are given the position in Cartesian components, so we can compute the velocity vector, using Eq. (2.16). The speed is then found by applying Eq. (2.9), i.e., by computing the magnitude of the velocity vector.

Computation Using Eq. (2.16), the velocity of P in each of the two frames is

$$\boxed{\vec{v}_{P/A}(t) = \dot{\vec{r}}_{P/A}(t) = 7.05t^2\,\hat{\imath}_A \text{ m/s},} \tag{5}$$

$$\boxed{\vec{v}_{P/B}(t) = \dot{\vec{r}}_{P/B}(t) = \left(6.39t^2\,\hat{\imath}_B + 2.98t^2\,\hat{\jmath}_B\right)\text{m/s}.} \tag{6}$$

As far as the speed is concerned, applying Eq. (2.9) to Eqs. (5) and (6), we obtain

$$\boxed{v_{P/A}(t) = 7.05t^2 \text{ m/s},} \tag{7}$$

$$\boxed{v_{P/B}(t) = \sqrt{\left(6.39t^2\right)^2 + \left(2.98t^2\right)^2}\text{ m/s} = 7.05t^2 \text{ m/s}.} \tag{8}$$

Helpful Information

Velocity revisited. Since the frames we are using do not move relative to one another, the velocity vector is the same no matter what frame is used. This is why it is correct to say that $\dot{\vec{r}}_{P/A} = \dot{\vec{r}}_{P/B}$ in Eqs. (5) and (6).

Discussion & Verification Since frames A and B do not move relative to each other, we expect the vectors in Eqs. (5) and (6) to be equal to each other, and therefore we expect to find the same value of speed in either frame. While it is not immediately obvious that Eqs. (5) and (6) are describing the same velocity vector, Eqs. (7) and (8) do confirm our expectation. In Example 2.3 we will show that $\vec{v}_{P/A} = \vec{v}_{P/B}$.

Part (c): Acceleration of P & Its Orientation

Road Map Using the velocity results in Eqs. (5) and (6), we can find the acceleration by using Eq. (2.17). We can then find the orientation of the acceleration vector relative to the trajectory by finding the angle between the tangent to the trajectory and the x axis and comparing that with the angle between the acceleration vector and the x axis. Since the velocity is tangent to the trajectory, we can find the angle between the trajectory and the x axis by finding the angle between the velocity and the x axis.

Computation Applying Eq. (2.17), the acceleration of P is given by

$$\boxed{\begin{aligned} \vec{a}_{P/A}(t) &= \dot{\vec{v}}_{P/A}(t) = 14.1t\,\hat{\imath}_A \text{ m/s}^2, \qquad (9)\\ \vec{a}_{P/B}(t) &= \dot{\vec{v}}_{P/B}(t) = (12.8t\,\hat{\imath}_B + 5.96t\,\hat{\jmath}_B) \text{ m/s}^2. \qquad (10)\end{aligned}}$$

Determining the orientation of $\vec{a}$ relative to the trajectory is not difficult in frame A since both $\vec{a}_{P/A}$ and $\vec{v}_{P/A}$ are always in the positive x_A direction. Therefore, in frame A, $\vec{a}$ is always tangent to the trajectory. In frame B, the angle between the trajectory and the x_B axis is given by

$$\theta_v = \tan^{-1}\left(\frac{v_y}{v_x}\right) = \tan^{-1}\left(\frac{2.98t^2}{6.39t^2}\right) = 25.0°. \qquad (11)$$

The angle between the acceleration and the x_B axis is

$$\boxed{\theta_a = \tan^{-1}\left(\frac{a_y}{a_x}\right) = \tan^{-1}\left(\frac{5.96t}{12.8t}\right) = 25.0°.} \qquad (12)$$

Since the angles in Eqs. (11) and (12) are equal, we conclude that the acceleration is tangent to the path, even when using frame B data.

Discussion & Verification Now that we found the acceleration in frames A and B in Eqs. (9) and (10), respectively, it is not obvious that these two vectors are the same. As with the velocity, we will show that this is so in Example 2.3. Regarding the orientation of $\vec{a}$ relative to the trajectory, we saw in frame A that $\vec{a}$ is always tangent to the trajectory. In frame B, Eqs. (11) and (12) tell us that the angle between $\vec{a}$ and the x_2 axis is always 25° and the angle between $\vec{v}$ and the x_2 axis is also always 25°. Therefore, we see that $\vec{a}$ is also always tangent to the trajectory in frame B. The fact that the accelerations in both frames are tangent to their respective trajectories gives us some confidence that the two accelerations we computed are the same and are correct.

Helpful Information

Choosing a frame of reference. An important criterion for choosing a frame of reference is *convenience*. Reference frames and (as we will see later in this chapter) coordinate systems profoundly impact how simply and directly the solution to a problem is obtained. Hence, a useful skill to cultivate is selection of the frame of reference and coordinate system that leads to the solution with greater ease.

Common Pitfall

Does Eq. (12) contradict what we said earlier about the acceleration not being tangent to the trajectory? Earlier in the section we stated that *in general* the acceleration is not tangent to the path. We also indicated that if the trajectory is curved, then we must expect the acceleration not to be parallel to the path. Therefore we can conclude that (1) there can be points along a path at which the acceleration is tangent to the path and that (2) at these points the path's curvature must be equal to zero. This is consistent with what we found in our example since the path in this example is a straight line, that is, *a line with no curvature*.

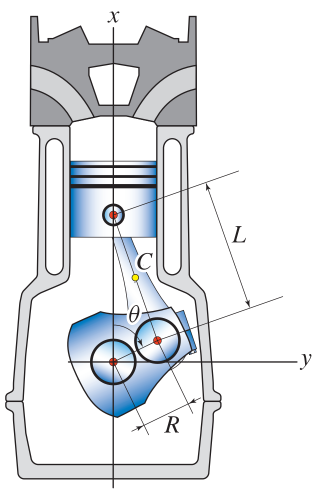

Figure P2.19–P2.21

Problems 2.19 through 2.21

Point C is a point on the connecting rod of a mechanism called a *slider-crank*. The x and y coordinates of C can be expressed as follows: $x_C = R\cos\theta + \frac{1}{2}\sqrt{L^2 - R^2\sin^2\theta}$ and $y_C = (R/2)\sin\theta$, where θ describes the position of the crank. If the crank rotates at a constant rate, then we can express θ as $\theta = \omega t$, where t is time and ω is the crank's angular velocity. Let $R = 0.1\,\text{m}$, $L = 0.25\,\text{m}$, and $\omega = 250\,\text{rad/s}$.

Problem 2.19 Find expressions for the velocity, speed, and acceleration of C.

Problem 2.20 Determine the maximum and minimum speeds of C as well as C's coordinates when the maximum and minimum speeds are achieved. In addition, determine the acceleration of C when the speed is at a minimum.

Problem 2.21 Plot the trajectory of point C for $0\,\text{s} < t < 0.025\,\text{s}$. For the same interval of time, plot the speed as a function of time as well as the components of the velocity and acceleration of C.

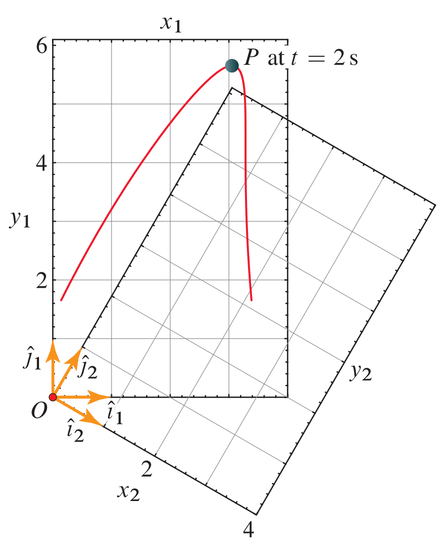

Figure P2.22

Problem 2.22

The motion of a point P with respect to Cartesian frames 1 and 2 is described by

$$(\vec{r}_{P/O})_1 = \left[(t + \sin t)\,\hat{\imath}_1 + \left(2 + 4t - t^2\right)\hat{\jmath}_1\right]\text{m}$$

and

$$(\vec{r}_{P/O})_2 = \Big\{\left[(t + \sin t)\cos\theta + \left(2 + 4t - t^2\right)\sin\theta\right]\hat{\imath}_2 + \left[-(t + \sin t)\sin\theta + \left(2 + 4t - t^2\right)\cos\theta\right]\hat{\jmath}_2\Big\}\,\text{m},$$

respectively, where t is time in seconds. Note that the two frames in this problem share the same origin, and therefore we are writing $(\vec{r}_{P/O})_1$ and $(\vec{r}_{P/O})_2$ to explicitly indicate that $(\vec{r}_{P/O})_1$ is expressed relative to frame 1 and $(\vec{r}_{P/O})_2$ is expressed relative to frame 2. Determine P's velocity and acceleration with respect to the two frames. In addition, determine the speed of P at time $t = 2\,\text{s}$, and verify that the speeds in the two frames are equal.

Problem 2.23

Let $\vec{r}_{P/A}$, $\vec{v}_{P/A}$, and $\vec{a}_{P/A}$ denote the position, velocity, and acceleration vectors of a point P with respect to the frame with origin at A. Let $\vec{r}_{P/B}$, $\vec{v}_{P/B}$, and $\vec{a}_{P/B}$ be the position, velocity, and acceleration vectors of the same point P with respect to the frame with origin at B. If frame B does not move relative to frame A, and if the frames are distinct, state whether or not each of the following relations is true and why.

(a) $\vec{r}_{P/A} - \vec{r}_{P/B} = \vec{0}$

(b) $\vec{v}_{P/A} - \vec{v}_{P/B} = \vec{0}$

(c) $\vec{v}_{P/A} \cdot \vec{a}_{P/B} = \vec{v}_{P/B} \cdot \vec{a}_{P/B}$

Note: Concept problems are about *explanations*, not computations.

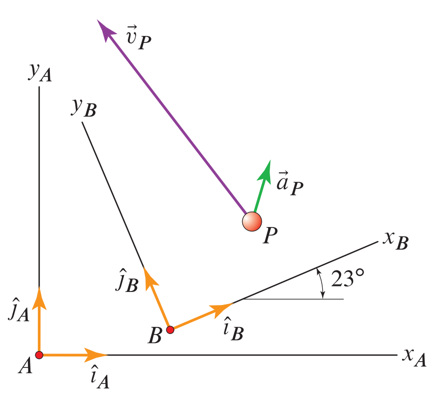

Figure P2.24

Problem 2.24

The velocity of point P relative to frame A is $\vec{v}_{P/A} = (-14.9\,\hat{\imath}_A + 19.4\,\hat{\jmath}_A)$ ft/s, and the acceleration of P relative to frame B is $\vec{a}_{P/B} = (3.97\,\hat{\imath}_B + 4.79\,\hat{\jmath}_B)$ ft/s^2. Knowing that frames A and B do not move relative to one another, determine the expressions for the velocity of P in frame B and the acceleration of P in frame A. Verify that the speed of P and the magnitude of P's acceleration are the same in the two frames.

2.2 Elementary Motions

This section examines in detail how to relate acceleration to position and velocity in a variety of situations found in applications. To better focus on how these relations are built, here we avoid dealing with vector quantities and we examine only one-dimensional motions.

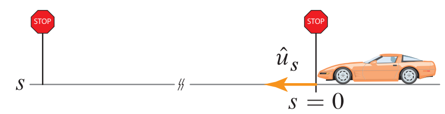

Figure 2.13
A car driving between two stop signs.

Driving down a city street

A car drives along a straight street between two stop signs (see Fig. 2.13). The car's velocity is given as

$$v = 9 - 9\cos\left(\tfrac{2}{5}t\right)\,\mathrm{m/s}, \quad 0\,\mathrm{s} \le t \le 5\pi\,\mathrm{s}, \tag{2.18}$$

which is plotted in Fig. 2.14. Given this information, we want to determine

1. The time it took to go from one stop sign to the other.
2. The distance between the two stop signs.
3. The acceleration at every instant along the way.

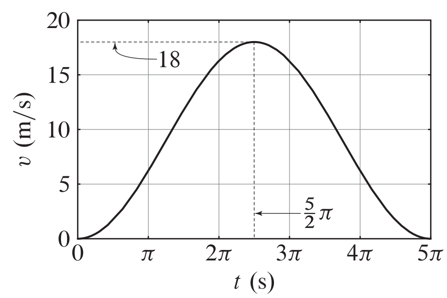

Figure 2.14
Velocity versus time curve for a car going between two stop signs.

We begin by observing that v in Eq. (2.18) is a scalar and therefore might not be used as a velocity, which is a vector. However, since the motion is one-dimensional, we can infer the direction of motion from the sign of v. Adopting this strategy and denoting the car's position by the coordinate s, we set

$$v = \dot{s}, \tag{2.19}$$

and by allowing v to take on both positive and negative values, we can then refer to v as the *velocity of the car*.

To answer question 1, since $v = 0$ at each of the stop signs, we can set to 0 the expression in Eq. (2.18) and solve for time, i.e.,

$$\begin{aligned} v = 9 - 9\cos\left(\tfrac{2}{5}t\right) = 0 \quad &\Rightarrow \quad \cos\left(\tfrac{2}{5}t\right) = 1 \\ \Rightarrow \quad \tfrac{2}{5}t = 0, 2\pi, 4\pi, \ldots \quad &\Rightarrow \quad t = 0, 5\pi, 10\pi\,\mathrm{s}, \ldots. \end{aligned} \tag{2.20}$$

Since the motion starts at $t = 0$, the two times of interest are $t_0 = 0\,\mathrm{s}$ and $t_1 = 5\pi\,\mathrm{s}$. Thus, the answer to question 1 is that it takes $t_1 - t_0 = 5\pi\,\mathrm{s} = 15.7\,\mathrm{s}$ to go from the first stop sign to the second.

To answer question 2, recall that $v = \dot{s} = ds/dt$ so that we can write $ds = v\,dt$ and then use *indefinite* integration to obtain

$$\int ds = \int v(t)\,dt = \int \left[9 - 9\cos\left(\tfrac{2}{5}t\right)\right] dt. \tag{2.21}$$

Alternatively, we can use *definite* integration to obtain

$$\int_0^{s(t)} ds = \int_0^t v(t)\,dt = \int_0^t \left[9 - 9\cos\left(\tfrac{2}{5}t\right)\right] dt, \tag{2.22}$$

where the lower limits of integration indicate that we have set the origin of the s axis to be the car's position at $t = 0$, and the upper limits indicate that we wish to express s as a function of time.

Helpful Information

Is there something wrong with Eq. (2.22)? To be rigorous, Eq. (2.22) should be written as

$$\begin{aligned} \int_0^{s(t)} d\sigma &= \int_0^t v(\tau)\,d\tau \\ &= \int_0^t \left[9 - 9\cos\left(\tfrac{2}{5}\tau\right)\right] d\tau, \end{aligned}$$

where the variables of integration are distinct from the variables used in the limits of integration. The symbols chosen for the variables of integration do not change the integral, and this is why the variables of integration in a definite integral are called *dummy variables*. However, we feel that it is more meaningful to *keep* the variables of integration as s and t in Eq. (2.22) to remind us of their physical significance. We will adopt such a practice throughout this text since the use of the same symbol for both the variables and the limits of integration will be clear from the context.

If the *acceleration is a constant* a_c, Eq. (2.29) becomes

$$\boxed{v = v_0 + a_c(t - t_0) \quad \text{(constant acceleration)},} \tag{2.41}$$

Eq. (2.31) becomes

$$\boxed{s = s_0 + v_0(t - t_0) + \tfrac{1}{2}a_c(t - t_0)^2 \quad \text{(constant acceleration)},} \tag{2.42}$$

and Eq. (2.38) becomes

$$\boxed{v^2 = v_0^2 + 2a_c(s - s_0) \quad \text{(constant acceleration)}.} \tag{2.43}$$

Circular motion and angular velocity

The relationships for rectilinear motion are applicable to any one-dimensional motion. To demonstrate this idea, we will now apply them to a common one-dimensional curvilinear motion: *circular motion*.

In Fig. 2.17 we see a particle A moving in a circle of radius r and center O. Since r is constant, the position of A can be described via a single coordinate such as the oriented arc length s or the angle θ. Again referring to Fig. 2.17, if the line OA rotates through the angle $\Delta\theta$ in the time Δt, then we can define an average time rate of change of the angle θ as $\omega_{\text{avg}} = \Delta\theta/\Delta t$. Hence, following the development of Section 2.1 and letting $\Delta t \to 0$, we obtain the instantaneous time rate of change of θ, i.e., $\dot{\theta}$, called the *angular velocity*, as

$$\boxed{\omega(t) = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta(t)}{dt} = \dot{\theta}(t).} \tag{2.44}$$

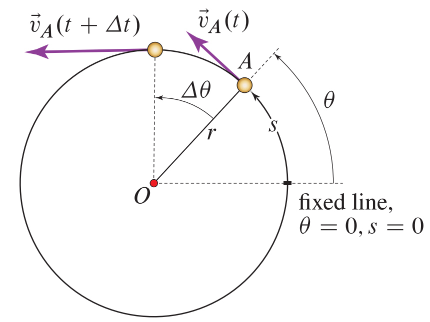

Figure 2.17
Particle A with speed $v_A = |\vec{v}_A|$ moving in a circle of radius r centered at O.

We can then define *angular acceleration* α by differentiating Eq. (2.44) with respect to time, i.e.,

$$\boxed{\alpha(t) = \frac{d\omega(t)}{dt} = \dot{\omega}(t) = \ddot{\theta}(t).} \tag{2.45}$$

When using the coordinate s, since $s = r\theta$ and r is constant, we can write

$$\boxed{\dot{s} = r\dot{\theta} = \omega r \quad \text{and} \quad \ddot{s} = r\ddot{\theta} = \alpha r.} \tag{2.46}$$

Circular motion relations

All of the relationships we developed for rectilinear motion apply equally well to circular motion, except that we need to replace the rectilinear variables by their circular counterparts. For example, Eq. (2.29) becomes

$$\omega(t) = \omega_0 + \int_{t_0}^{t} \alpha(t)\, dt. \tag{2.47}$$

Table 2.2 lists each kinematic variable in rectilinear motion and the corresponding kinematic variable for circular motion. Replacing each rectilinear motion variable with its circular motion counterpart in Eqs. (2.29)–(2.43), we obtain the corresponding circular motion equations. Finally, if the angular acceleration is constant, we can use the constant acceleration relations with a_c replaced by α_c.

Table 2.2
Correspondence of kinematic variables between rectilinear and circular motion.

Kinematic variable	Rectlinear motion	Circular motion[a]
time	t	t
position	s	θ
velocity	v	ω
acceleration	a	α

[a] Except for time, each of these should have the word *angular* in front of its kinematic variable name.

End of Section Summary

In this section we have developed relationships linking a single coordinate and its time derivatives. These relations have been categorized based on how the primary piece of information is provided:

1. If the acceleration is provided as a function of time, i.e., $a = a(t)$, for velocity and position, we have

Eqs. (2.29) and (2.31), p. 57

$$v(t) = v_0 + \int_{t_0}^{t} a(t)\,dt,$$

$$s(t) = s_0 + v_0(t - t_0) + \int_{t_0}^{t}\left[\int_{t_0}^{t} a(t)\,dt\right]dt.$$

2. If the acceleration is provided as a function of velocity, i.e., $a = a(v)$, for time and position, we have

Eq. (2.33), p. 57, and Eq. (2.36), p. 58

$$t(v) = t_0 + \int_{v_0}^{v} \frac{1}{a(v)}\,dv,$$

$$s(v) = s_0 + \int_{v_0}^{v} \frac{v}{a(v)}\,dv.$$

3. If the acceleration is provided as a function of position, i.e., $a = a(s)$, for velocity and time, we have

Eqs. (2.38) and (2.40), p. 58

$$v^2(s) = v_0^2 + 2\int_{s_0}^{s} a(s)\,ds,$$

$$t(s) = t_0 + \int_{s_0}^{s} \frac{ds}{v(s)}.$$

4. If the acceleration is a constant a_c, for velocity and position, we have

Eqs. (2.41)–(2.43), p. 59

$$v = v_0 + a_c(t - t_0),$$

$$s = s_0 + v_0(t - t_0) + \tfrac{1}{2}a_c(t - t_0)^2,$$

$$v^2 = v_0^2 + 2a_c(s - s_0).$$

Circular motion. For circular motion, the equations summarized in items 1–4 above hold as long as we use the replacement rules

$$s \to \theta, \quad v \to \omega, \quad a \to \alpha,$$

where $\omega = \dot{\theta}$ and $\alpha = \ddot{\theta}$ are the *angular velocity* and *angular acceleration*, respectively.

EXAMPLE 2.8 Measuring the Depth of a Well by Relating Time, Velocity, and Acceleration

We can estimate the depth of a well by measuring the time it takes for a rock dropped from the top of the well to reach the water below. Assuming that gravity is the only force acting on the rock, estimate a well's depth under two different assumptions: the speed of sound is (a) finite and equal to $v_s = 340\,\mathrm{m/s}$ and (b) infinite. Also, compare the two estimates to provide a "rule of thumb" as to when we can assume that the speed of sound is infinite.

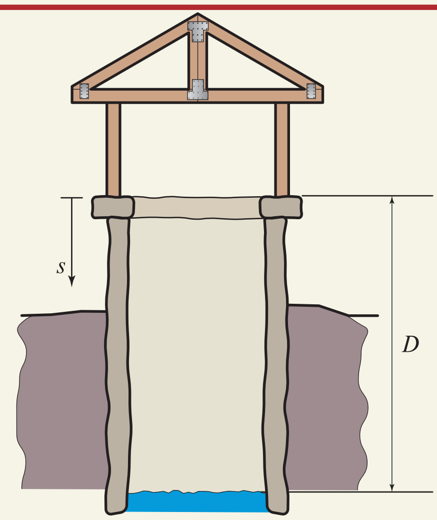

Figure 1
A well of depth D showing the positive direction of the coordinate s.

SOLUTION

Road Map In this problem our time measure is the sum of two parts: (1) the time taken by the rock to go from the top to the bottom of the well and (2) the time taken by sound to go from the bottom to the top of the well. The well's depth can be related to the first time by assuming that the rock travels at a constant acceleration, namely, $g = 9.81\,\mathrm{m/s^2}$. The well's depth can also be related to the second time by assuming that sound travels at a constant speed, which will be assumed to be finite in Part (a) and infinite in Part (b). By requiring that the two depth estimates be identical, we will be able to find a relation between the well's depth and the overall measured time.

Part (a): Finite Sound Speed

Computation Figure 1 shows a well of unknown depth D. Let t_m, t_i, and t_s be the (total) measured time, the time taken by the rock to fall the distance D and impact with the water, and the time it takes sound to go back up, respectively, so that

$$t_m = t_i + t_s. \tag{1}$$

The motion of the rock falling the distance D is a rectilinear motion with constant acceleration $g = 9.81\,\mathrm{m/s^2}$. Hence, by choosing a coordinate axis pointing from the top to the bottom of the well, noting that the rock starts at $s_0 = 0\,\mathrm{m}$, and assuming that the rock is released with initial velocity v_0 equal to zero, Eq. (2.42) tells us that

$$D = \tfrac{1}{2} g t_i^2 \quad \Rightarrow \quad t_i = \sqrt{\frac{2D}{g}}. \tag{2}$$

As soon as the rock hits the water, a sound wave traveling with a constant velocity $v_s = 340\,\mathrm{m/s}$, and therefore with constant acceleration $a_s = 0\,\mathrm{m/s^2}$, goes from the bottom of the well up to the observer's ear at $s = 0$. Equation (2.42) then tells us that

$$0 = D - v_s(t_m - t_i) \quad \Rightarrow \quad D = v_s t_s \quad \Rightarrow \quad t_s = \frac{D}{v_s}, \tag{3}$$

where we have used Eq. (1) to write $t_s = t_m - t_i$. Next, using the expressions for t_i and t_s in Eqs. (2) and (3), respectively, Eq. (1) becomes

$$t_m = \sqrt{\frac{2D}{g}} + \frac{D}{v_s}. \tag{4}$$

This equation can be solved for D to obtain (see the Helpful Information note in the margin for details)

$$\boxed{D = v_s t_m - \frac{v_s^2}{g}\left(\sqrt{1 + \frac{2 t_m g}{v_s}} - 1\right).} \tag{5}$$

> **Helpful Information**
>
> **Solving Eq. (4) for D.** To solve Eq. (4) for D, we first rewrite it so as to isolate the square root term, i.e.,
>
> $$D - v_s t_m = -v_s \sqrt{2D/g}.$$
>
> We then square each side to obtain
>
> $$D^2 - 2D v_s t_m + v_s^2 t_{\mathrm{m}}^2 = v_s^2 \frac{2D}{g},$$
>
> which can be rearranged to read
>
> $$D^2 - 2v_s\left(t_m + \frac{v_s}{g}\right) D + v_s^2 t_{\mathrm{m}}^2 = 0.$$
>
> This is a quadratic equation in D with the following two roots:
>
> $$D = v_s t_m + \frac{v_s^2}{g}\left(1 \pm \sqrt{1 + \frac{2 t_m g}{v_s}}\right).$$
>
> Only one of these roots is physically meaningful. The solution with the plus sign in front of the square root term yields a nonzero value for D when $t_m = 0$. This result contradicts Eq. (4), so the only acceptable solution is the one with the minus sign.

EXAMPLE 2.9 *Acceleration Function of Velocity: Descent of a Skydiver*

The skydiver shown in Figs. 1 and 2 has deployed his parachute after free-falling at $v_0 = 44.5\,\mathrm{m/s}$. We will learn how to derive the governing equations for systems such as this in Chapter 3. For now, it suffices to say that the relevant forces on the skydiver are his total weight (i.e., his body weight and that of his equipment) and the drag force due to the parachute. If we model the drag force as being proportional to the square of the skydiver's velocity, or $F_d = C_d v^2$, where C_d denotes a drag coefficient,* Newton's second law tells us that the skydiver's acceleration is $a = g - C_d v^2/m$. Letting $C_d = 43.2\,\mathrm{kg/m}$, $m = 110\,\mathrm{kg}$, and $g = 9.81\,\mathrm{m/s^2}$, determine

(a) The skydiver's velocity as a function of time.

(b) The terminal velocity reached by the skydiver.

(c) The skydiver's position as a function of time.

Figure 1
A skydiver descending.

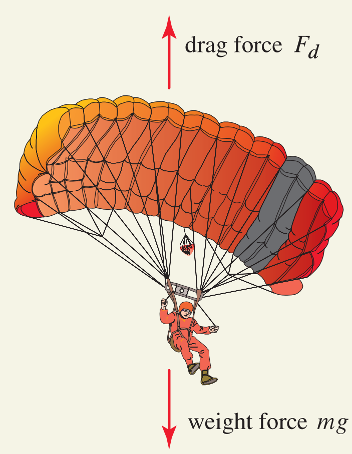

Figure 2
Skydiver with drag and weight forces depicted.

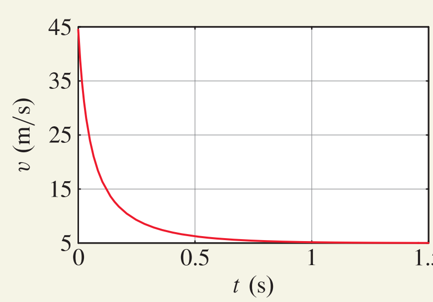

Figure 3
Velocity of the skydiver as he decends with his parachute deployed.

SOLUTION

Part (a): From Acceleration to Velocity

Road Map Since the acceleration is not given as a function of time, but as a function of velocity, i.e., $a = a(v)$, we cannot obtain $v(t)$ by integrating a with respect to time. However, recalling that $a = dv/dt$ can be rewritten as $dt = dv/a$, we can obtain time as a function of velocity, i.e., $t = t(v)$ and then we will try to invert this relationship to obtain $v = v(t)$. This is the strategy followed in developing Eq. (2.33) for the case when $a = a(v)$.

Computation We begin by applying Eq. (2.33), or, equivalently, rewriting $a = dv/dt$ as $dt = dv/a(v)$ and integrating both sides to obtain

$$t(v) = \int_{v_0}^{v} \frac{dv}{g - C_d v^2/m}$$
$$= -\frac{1}{2}\sqrt{\frac{m}{gC_d}}\ln\left[\left(\frac{v\sqrt{C_d} - \sqrt{mg}}{v\sqrt{C_d} + \sqrt{mg}}\right)\left(\frac{v_0\sqrt{C_d} + \sqrt{mg}}{v_0\sqrt{C_d} - \sqrt{mg}}\right)\right], \tag{1}$$

where we have set $v = v_0$ for $t = 0$, and where we note that this integral can be obtained using a comprehensive table of integrals† or a software package such as Mathematica. We now have $t(v)$, but we want $v(t)$. Hence, to invert Eq. (1), we first multiply both sides by $-2\sqrt{gC_d/m}$ and then exponentiate both sides to obtain

$$e^{-2t\sqrt{\frac{gC_d}{m}}} = \left(\frac{v\sqrt{C_d} - \sqrt{mg}}{v\sqrt{C_d} + \sqrt{mg}}\right)\left(\frac{v_0\sqrt{C_d} + \sqrt{mg}}{v_0\sqrt{C_d} - \sqrt{mg}}\right). \tag{2}$$

Solving Eq. (2) for v and simplifying, we obtain

$$\boxed{v(t) = \sqrt{\frac{mg}{C_d}}\,\frac{v_0\sqrt{C_d} + \sqrt{mg} + \left(v_0\sqrt{C_d} - \sqrt{mg}\right)e^{-2t\sqrt{\frac{gC_d}{m}}}}{v_0\sqrt{C_d} + \sqrt{mg} - \left(v_0\sqrt{C_d} - \sqrt{mg}\right)e^{-2t\sqrt{\frac{gC_d}{m}}}},} \tag{3}$$

the plot of which can be found in Fig. 3 for the parameters given. Notice that the skydiver starts out at $44.5\,\mathrm{m/s}$ at $t = 0\,\mathrm{s}$ and quickly (in about one second) slows down to approximately $5\,\mathrm{m/s}$.

---

* The drag coefficient used in this problem is a condensed version of the drag coefficient found in Example 2.8, and they are related according to $C_d = \frac{1}{2}C_D\rho A$.

† See, for example, A. Jeffrey, *Handbook of Mathematical Formulas and Integrals*, 3rd ed., Academic Press, 2003; or R. J. Tallarida, *Pocket Book of Integrals and Mathematical Formulas*, 3rd ed., CRC Press, Boca Raton, FL, 1999. In addition, there are several Internet resources such as <http://en.wikipedia.org/wiki/Lists_of_integrals>.

Problem 2.71

Figure P2.71

Approximately 1 h 15 min into the movie "King Kong" (the one directed by Peter Jackson), there is a scene in which Kong is holding Ann Darrow (played by the actress Naomi Watts) in his hand while swinging his arm in anger. A quick analysis of the movie indicates that at a particular moment Kong displaces Ann from rest by roughly 10 ft in a span of four frames. Knowing that the DVD plays at 24 frames per second and assuming that Kong subjects Ann to a constant acceleration, determine the acceleration Ann experiences in the scene in question. Express your answer in terms of the acceleration due to gravity g. Comment on what would happen to a person *really* subjected to this acceleration.

Problem 2.72

Derive the constant acceleration relation in Eq. (2.41), starting from Eq. (2.33). State what assumption you need to make about the acceleration a to complete the derivation. Finally, use Eq. (2.36), along with the result of your derivation, to derive Eq. (2.42). Be careful to do the integral in Eq. (2.36) before substituting your result for $v(t)$ (try it without doing so, to see what happens). After completing this problem, notice that Eqs. (2.41) and (2.42) are *not* subject to the same assumption you needed to make to solve both parts of this problem.

Problems 2.73 through 2.75

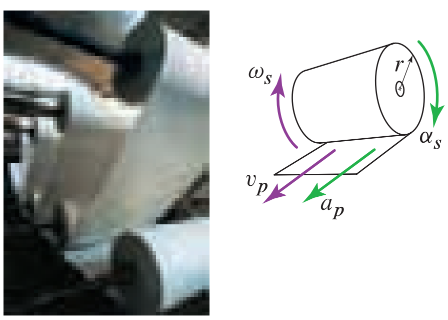

Figure P2.73–P2.75

The spool of paper used in a printing process is unrolled with velocity v_p and acceleration a_p. The thickness of the paper is h, and the outer radius of the spool at any instant is r.

Problem 2.73 If the velocity at which the paper is unrolled is *constant*, determine the angular acceleration α_s of the spool as a function of r, h, and v_p. Evaluate your answer for $h = 0.0048$ in., for $v_p = 1000$ ft/min, and two values of r, that is, $r_1 = 25$ in. and $r_2 = 10$ in.

Problem 2.74 If the velocity at which the paper is unrolled is *not constant*, determine the angular acceleration α_s of the spool as a function of r, h, v_p, and a_p. Evaluate your answer for $h = 0.0048$ in., $v_p = 1000$ ft/min, $a_p = 3\,\text{ft/s}^2$, and two values of r, that is, $r_1 = 25$ in. and $r_2 = 10$ in.

Problem 2.75 If the velocity at which the paper is unrolled is *constant*, determine the angular acceleration α_s of the spool as a function of r, h, and v_p. Plot your answer for $h = 0.0048$ in. and $v_p = 1000$ ft/min as a function of r for 1 in. $\leq r \leq 25$ in. Over what range does α_s vary?

basis. Specifically, applying Eq. (2.42) (on p. 59) in the x direction yields

$$x(t) = x(0) + \dot{x}(0)(t - t_0) + \tfrac{1}{2}0\,(t - t_0)^2 \quad \Rightarrow \quad x(t) = v_0 \cos\beta\, t, \quad (2.50)$$

where $x(0) = 0$, $\dot{x}(0) = v_0 \cos\beta$, and $t_0 = 0$ (see Fig. 2.19). Similarly, for the y direction we have

$$y(t) = y(0) + \dot{y}(0)(t - t_0) - \tfrac{1}{2}g(t - t_0)^2 \quad \Rightarrow \quad y(t) = v_0 \sin\beta\, t - \tfrac{1}{2}gt^2, \quad (2.51)$$

where $y(0) = 0$ and $\dot{y}(t) = v_0 \sin\beta$. We can now obtain the pumpkin's trajectory by eliminating time from Eqs. (2.50) and (2.51). Solving for t in Eq. (2.50) gives $t = x/(v_0 \cos\beta)$. Substituting this expression into Eq. (2.51) and simplifying, we have

$$y = (\tan\beta)x - \left(\frac{g \sec^2\beta}{2v_0^2}\right)x^2, \quad (2.52)$$

which shows that the trajectory of the pumpkin is a parabola!

The simplicity of the pumpkin's motion results from studying the motion projectile without accounting for air resistance and the dependence of gravity on height. We now proceed to formally define such a motion and make a few remarks about its analysis.

Interesting Fact

Engineers and trebuchets. Trebuchets were often called *engines* in Europe (from the Latin *ingenium*, or "an ingenious contrivance"). The people who designed, made, and used trebuchets were called *ingeniators*, and it is from this that we derive our modern terms *engineer* and *engineering*. See P. E. Chevedden, L. Eigenbrod, V. Foley, and W. Soedel, "The Trebuchet," *Scientific American*, **273**(1), pp. 66–71, July 1995, for an article about trebuchets.

Projectile motion

We define *projectile motion* as a motion in which the acceleration is *constant* and given by

$$a_{\text{horiz}} = 0 \quad \text{and} \quad a_{\text{vert}} = -g, \quad (2.53)$$

where, referring to Fig. 2.22, a_{horiz} and a_{vert} are the components of the acceleration in the horizontal and vertical directions, respectively, and where we have chosen the vertical direction to be positive *upward*. Our definition of projectile motion is a very simplified description of true projectile motion because the relations in Eq. (2.53) neglect air resistance and changes in gravitational attraction with changes in height. These effects will be considered in Chapters 3 and 5, respectively.

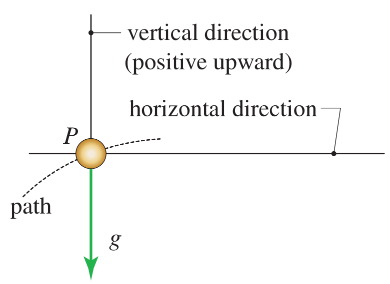

Figure 2.22
Acceleration of a point P in projectile motion.

To describe the velocity, position, and trajectory of a projectile, we typically use a Cartesian coordinate system with axes parallel and perpendicular to the direction of gravity, as was done earlier in the analysis of the pumpkin problem. However, other choices are possible and, in some cases, more convenient (for a description of projectile motion using general Cartesian coordinate system see Example 2.13).

In deriving Eq. (2.52), we showed that the trajectory of a projectile is a parabola. This result is independent of coordinate system. However, the expression of the parabola does depend on the coordinate system used. The trajectory is of the form derived in Eq. (2.52), i.e., of the form $y = C_0 + C_1 x + C_2 x^2$ (C_0, C_1, and C_3 are constant coefficients),* if the y axis of our Cartesian coordinate system is parallel to the direction of gravity (see Example 2.13 for a case in which the y axis is not parallel to gravity).

Helpful Information

Trajectory of a projectile. The trajectory of a projectile is a parabola, and its mathematical expression is of the form $y = C_0 + C_1 x + C_2 x^2$ only when using a Cartesian coordinate system with the y axis parallel to the direction of gravity.

* In Eq. (2.52) we found $C_0 = 0$, $C_1 = \tan\beta$, and $C_2 = -(g \sec^2\beta)/(2v_0^2)$.

which means that for each value of y_B, there are two possible values of the firing angle θ: θ_1 and θ_2. Substituting in values for all constants, including the two different values for y_B, we obtain

$$\boxed{\begin{aligned} y_B &= 4\,\text{ft} \quad \Rightarrow \quad \begin{cases} \theta_1 = 0.991678°, \\ \theta_2 = 89.2375041°, \end{cases} \qquad (8) \\ y_B &= 7\,\text{ft} \quad \Rightarrow \quad \begin{cases} \theta_1 = 1.163590°, \\ \theta_2 = 89.2374736°. \end{cases} \qquad (9) \end{aligned}}$$

Equations (8) and (9) give us the values of θ needed to hit the bottom and top of the sign, respectively. Referring to Eq. (8), it is probably intuitive that if we chose $\theta_1 < \theta < \theta_2$, we would overshoot the bottom of the target whereas we would undershoot it for $\theta < \theta_1$ and $\theta > \theta_2$. For example, substituting $\theta = 45°$ (i.e., a value of θ between those in Eq. (8)) into Eq. (4) gives $y_B = 26.6$ ft, which, as expected, is larger than 4 ft. Extending the discussion to Eq. (9), we can then say that if $\theta_1 < \theta < \theta_2$ in Eq. (9), we would overshoot the top of the sign whereas we would undershoot it for $\theta < \theta_1$ and $\theta > \theta_2$. We can therefore conclude that there are two ranges of firing angles such that our projectile will hit the target and that these ranges are given by

$$\boxed{\begin{aligned} &0.991678° \le \theta \le 1.163590° \qquad (10) \\ \text{and} \qquad &89.23747360° \le \theta \le 89.23750406°. \qquad (11) \end{aligned}}$$

Helpful Information

Number of digits in calculations. In the numerical calculations shown in this example, the differences in some of the numbers are *so* small that we are keeping many more digits than we normally would. If we did not, the differences would not be apparent.

Discussion & Verification In obtaining the angle ranges in Eqs. (10) and (11) we went through a simple verification step by computing the answer for $\theta = 45°$, in which we saw that our results were as expected. To extend our discussion, let's now consider the size of the ranges in Eqs. (10) and (11):

$$\Delta\theta_1 = 1.163590° - 0.991678° = 0.171912°, \qquad (12)$$
$$\Delta\theta_2 = 89.23747360° - 89.23750406° = -0.00003046°. \qquad (13)$$

Equations (10) and (12) tell us that to hit our target, we need to elevate our launcher about 1° with an accuracy of 0.17°. Equations (11) and (13) tell us that we can also hit the target if we elevate our launcher to approximately 89.2°, but this time we have an *extremely small* margin of error. In fact, we need to be accurate to within three one-hundred thousandths of a degree! In addition, we also need to keep in mind that our model does not account for aerodynamic effects, which place an additional accuracy burden on our aim. These considerations tell us that, in practical applications, we need to rely on an *active guidance system* rather than the accuracy of the launch angles.

Let's complete this example by comparing the angle $\Delta\theta_1$ with the angle subtended by the target at a distance of 1000 ft. Referring to Fig. 2, we can see that the angle subtended is given by

$$\beta = \gamma - \phi, \qquad (14)$$

where

$$\gamma = \tan^{-1}\left(\frac{7}{1000}\right) = 0.401064° \quad \text{and} \quad \phi = \tan^{-1}\left(\frac{4}{1000}\right) = 0.229182°, \qquad (15)$$

so that $\beta = 0.171882°$. Since β is very close to the $\Delta\theta_1$ in Eq. (12) and since computing β is simpler than computing $\Delta\theta_1$, we might think that we could have computed β to approximate $\Delta\theta_1$. However, in general, the elevation angle ranges and angle subtended by the target can be substantially different.

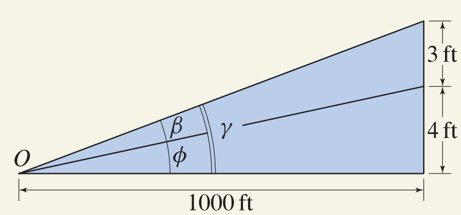

Figure 2

Not drawn to scale—the vertical dimension has been greatly exaggerated so that the angles can be easily seen.

EXAMPLE 2.12 *Initial Speed and Elevation Angle of a Projectile*

A baseball batter makes contact with a ball about 4 ft above the ground and hits it hard enough that it *just* clears the center field wall, which is 400 ft away and is 9 ft high. How fast must the ball be moving and at what angle must it be hit so that it just clears the center field wall as shown in Fig. 1?

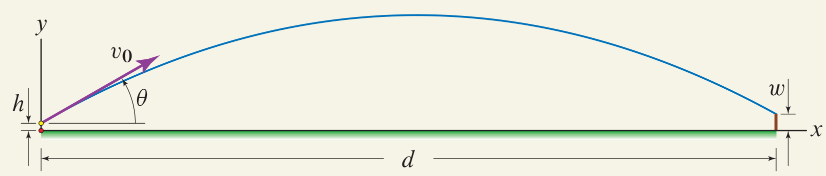

Figure 1. Side view, drawn to scale, of the given baseball field with all parameters defined. In the trajectory shown, the baseball was hit 4 ft off the ground, at 123.2 ft/s, and at a 30° angle so that it *just* clears a 9 ft fence that is 400 ft away.

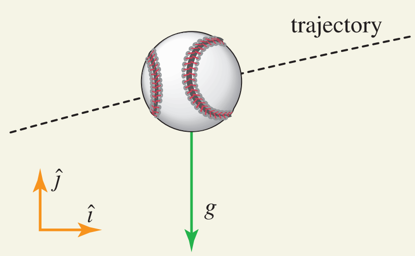

Figure 2
The only nonzero component of acceleration of the baseball.

SOLUTION

Road Map Referring to Fig. 2, we model the ball as a projectile with acceleration given by $a_x = 0$ and $a_y = -g$. We know the starting and ending locations of the projectile, and we wish to determine the v_0 and θ required to get it from start to finish. Therefore, we can proceed as in Example 2.11, i.e., by writing the projectile's x and y positions as a function of time and then, eliminating time, we will obtain an expression for v_0 in terms of θ.

Computation Since both components of acceleration are constant, we can apply the constant acceleration equation, Eq. (2.42) (on p. 59), in both the x and y directions to obtain

$$x = x_0 + v_{0x}t \quad \Rightarrow \quad d = 0 + v_0 t \cos\theta, \tag{1}$$

$$y = y_0 + v_{0y}t + \tfrac{1}{2}a_y t^2 \quad \Rightarrow \quad w = h + v_0 t \sin\theta - \tfrac{1}{2}g t^2. \tag{2}$$

Equations (1) and (2) are two equations for the three unknowns v_0, θ, and t. Since we are interested in v_0 and θ, we can eliminate t from these two equations and then solve for v_0 as a function of θ to obtain

$$\boxed{v_0 = d\sqrt{\frac{g}{2\cos\theta\,[(h-w)\cos\theta + d\sin\theta]}}.} \tag{3}$$

This result tells us that there are infinitely many combinations of v_0 and θ that will *just* get the baseball over the center field fence.

Discussion & Verification The solution to the problem is an expression rather than a specific quantitative answer. To verify that Eq. (3) is correct, we first check that its dimensions are correct. Letting L and T denote dimensions of length and time, we have $[g] = L/T^2$ and $[h] = [w] = [d] = L$. Therefore, the dimensions of the argument of the square root in Eq. (3) are T^{-2}, so that the overall dimensions of the right-hand side of Eq. (3) are L/T, as expected. Further verification requires that we study the behavior of the expression in Eq. (3), as is done next.

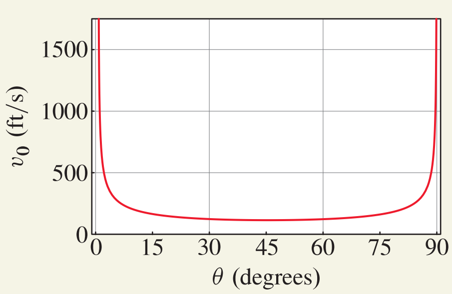

Figure 3
Plot of Eq. (3), that is, v_0 as a function of θ.

A Closer Look For $d = 400$ ft, $h = 4$ ft, $w = 9$ ft, and $g = 32.2\,\text{ft/s}^2$, the plot of the required v_0 for $0° \le \theta \le 90°$ is shown in Fig. 3. By careful inspection of the left side of the curve, we see that the curve approaches an asymptote value of θ other than

EXAMPLE 2.13 *Projectile Motion in a General Cartesian Coordinate System*

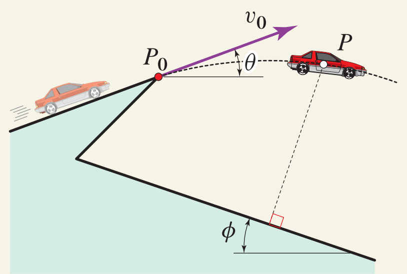

Figure 1
A movie stunt with a car jumping off a ridge.

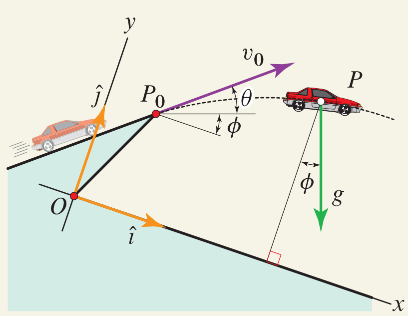

Figure 2
Acceleration in relation to the given coordinate system.

As part of a movie stunt, a car P runs off a ridge at the point P_0 with a speed v_0 as shown. Derive expressions for the velocity, position, and trajectory of the car such that it is easy to keep track of the perpendicular distance between the car and the lower incline.

SOLUTION

Road Map We model the car's motion as a projectile motion. By choosing the Cartesian coordinate system in Fig. 2, the perpendicular distance between the car and the incline is directly provided by the y coordinate of the point P. Therefore, we will solve the problem by using the coordinate system shown. We begin by determining the components of the acceleration in the x and y directions. The x and y components of the acceleration are constant so that we can obtain velocity and position by direct application of constant acceleration equations. Finally, the trajectory is found by eliminating time from the description of the position.

Computation Given the orientation of the y axis relative to gravity, we have

$$a_x = \ddot{x} = g \sin\phi \quad \text{and} \quad a_y = \ddot{y} = -g\cos\phi. \tag{1}$$

Since a_x and a_y are constant, the velocity components of P can be obtained as functions of time via a direct application of Eq. (2.41) on p. 59, i.e.,

$$\boxed{\begin{aligned} v_x(t) &= v_0\cos(\theta+\phi) + (g\sin\phi)(t-t_0), \\ v_y(t) &= v_0\sin(\theta+\phi) - (g\cos\phi)(t-t_0), \end{aligned}} \qquad \begin{aligned}(2)\\(3)\end{aligned}$$

where t_0 is the time at which P is at P_0 and where $v_0\cos(\theta+\phi)$ and $v_0\sin(\theta+\phi)$ are the x and y components of the initial velocity, respectively.

By direction application of Eq. (2.42) on p. 59, the position of P is given by

$$\boxed{\begin{aligned} x(t) &= x_0 + v_0\cos(\theta+\phi)(t-t_0) + \tfrac{1}{2}(g\sin\phi)(t-t_0)^2, \\ y(t) &= y_0 + v_0\sin(\theta+\phi)(t-t_0) - \tfrac{1}{2}(g\cos\phi)(t-t_0)^2, \end{aligned}} \qquad \begin{aligned}(4)\\(5)\end{aligned}$$

where x_0 and y_0 are the coordinates of the point P_0.

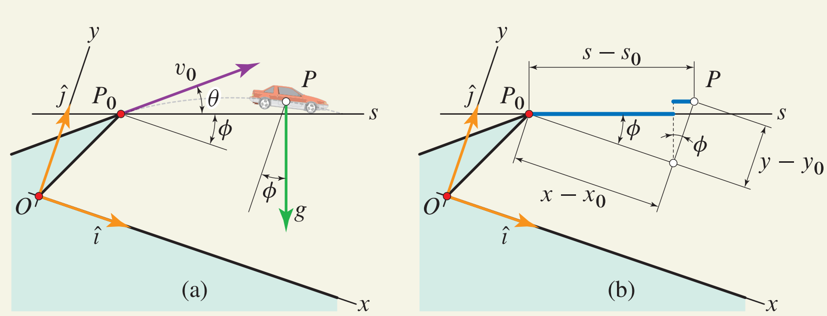

Figure 3. (a) Horizontal direction s and (b) displacements $x - x_0$ and $y - y_0$ in the x and y directions corresponding to the horizontal displacement $s - s_0$. The length of the blue segment to the left of the dotted line is $(x - x_0)\cos\phi$, and the length of the blue segment to the right of the dotted line is $(y - y_0)\sin\phi$.

PROBLEMS

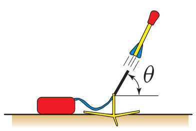

Figure P2.77

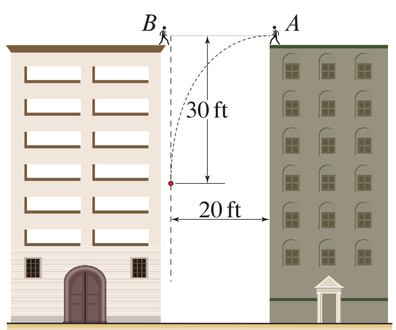

Figure P2.78

Problem 2.76

The discussion in Example 2.12 revealed that the angle θ had to be greater than $\theta_{\min} = 0.716°$. Find an analytical expressions for $\theta_{\min}$ in terms of h, w, and d.

Problem 2.77

A stomp rocket is a toy consisting of a hose connected to a "blast pad" (i.e., an air bladder) at one end and to a short pipe mounted on a tripod at the other end. A rocket with a hollow body is mounted onto the pipe and is propelled into the air by "stomping" on the blast pad. Some manufactures claim that one can shoot a rocket over 200 ft in the air. Neglecting air resistance, determine the rocket's minimum initial speed such that it reaches a maximum flight height of 200 ft.

Problem 2.78

Stuntmen A and B are shooting a movie scene in which A needs to pass a gun to B. Stuntman B is supposed to start falling vertically precisely when A throws the gun to B. Treating the gun and the stuntman B as particles, find the velocity of the gun as it leaves A's hand so that B will catch it after falling 30 ft.

Problem 2.79

The jaguar A leaps from O at speed $v_0 = 6\,\text{m/s}$ and angle $\beta = 35°$ relative to the incline to try to intercept the panther B at C. Determine the distance R that the jaguar jumps from O to C (i.e., R is the distance between the two points of the trajectory that intersect the incline), given that the angle of the incline is $\theta = 25°$.

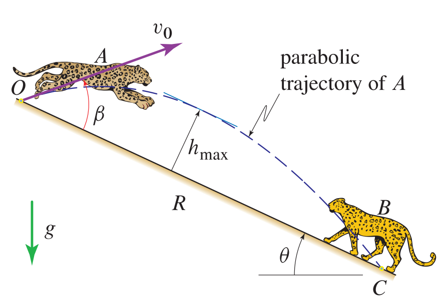

Figure P2.79

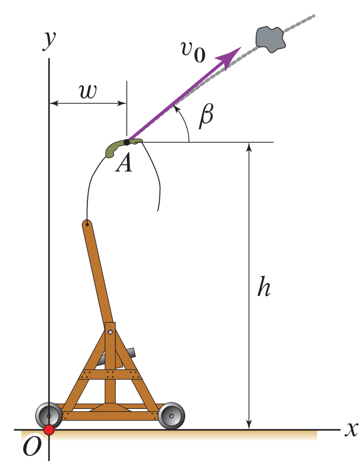

Figure P2.80

Problem 2.80

If the projectile is released at A with initial speed v_0 and angle β, derive the projectile's trajectory, using the coordinate system shown. Neglect air resistance.

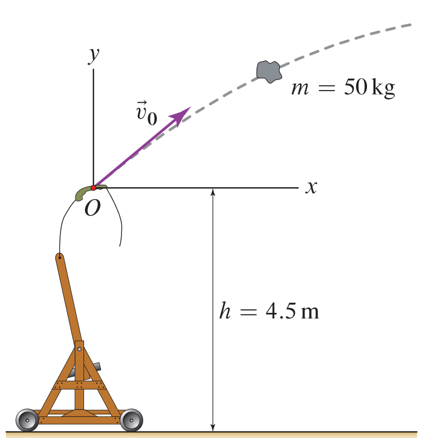

Figure P2.81

Problem 2.81

A trebuchet releases a rock with mass $m = 50\,\text{kg}$ at point O. The initial velocity of the projectile is $\vec{v}_0 = (45\,\hat{\imath} + 30\,\hat{\jmath})\,\text{m/s}$. Neglecting aerodynamic effects, determine where the rock will land and its time of flight.

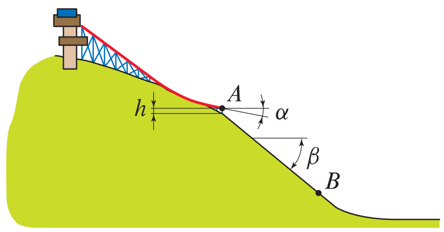

Figure P2.88

Problem 2.88

An alpine ski jumper can fly distances in excess of 100 m* by using his or her body and skis as a "wing" and therefore taking advantage of aerodynamic effects. With this in mind and assuming that a ski jumper could survive the jump, determine the distance the jumper could "fly" without aerodynamic effects, i.e., if the jumper were in free fall after clearing the ramp. For the purpose of your calculation, use the following typical data: $\alpha = 11°$ (slope of ramp at takeoff point A), $\beta = 36°$ (average slope of the hill),† $v_0 = 86\,\text{km/h}$ (speed at A), $h = 3\,\text{m}$ (height of takeoff point with respect to the hill). Finally, for simplicity, let the jump distance be the distance between the takeoff point A and the landing point B.

Problems 2.89 and 2.90

A soccer player practices kicking a ball from A directly into the goal (i.e., the ball does not bounce first) while clearing a 6 ft tall fixed barrier.

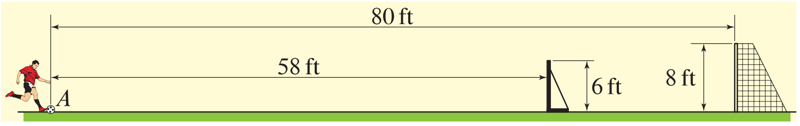

Figure P2.89 and P2.90

Problem 2.89 Determine the minimum speed that the player needs to give the ball to accomplish the task. *Hint:* To find $(v_0)_{\text{min}}$, consider the equation for the projectile's trajectory (see, e.g., Eq. (2.52)) for the case in which the ball reaches the goal at its base. Then solve this equation for the initial speed v_0 as a function of the initial angle θ, and finally find $(v_0)_{\text{min}}$ as you learned in calculus. Don't forget to check whether or not the ball clears the barrier.

Problem 2.90 Find the initial speed and angle that allow the ball to barely clear the barrier while barely reaching the goal at its base. *Hint:* As shown in Eq. (2.52), a projectile's trajectory can be given the form $y = C_1 x - C_2 x^2$ where the coefficients C_1 and C_2 can be found by forcing the parabola to go through two given points.

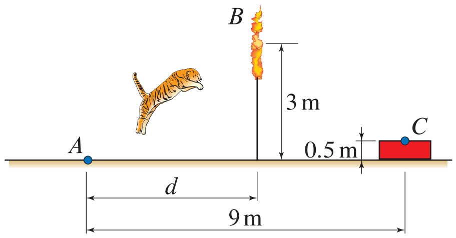

Figure P2.91 and P2.92

Problems 2.91 and 2.92

In a circus act a tiger is required to jump from point A to point C so that it goes through the ring of fire at B. *Hint:* As shown in Eq. (2.52), a projectile's trajectory can be given the form $y = C_1 x - C_2 x^2$ where the coefficients C_1 and C_2 can be found by forcing the parabola to go through two given points.

Problem 2.91 Determine the tiger's initial velocity if the ring of fire is placed at a distance $d = 5.5\,\text{m}$ from A. Furthermore, determine the slope of the tiger's trajectory as the tiger goes through the ring of fire.

Problem 2.92 Determine the tiger's initial velocity as well as the distance d so that the slope of the tiger's trajectory as the tiger goes through the ring of fire is completely horizontal.

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* On March 20, 2005, using the very large ski ramp at Planica, Slovenia, Bjørn Einar Romøren of Norway set the world record by flying a distance of 239 m.

† While the given average slope of the landing hill is accurate, you should know that, according to regulations, the landing hill must have a curved profile. Here, we have chosen to use a landing hill with a *constant* slope of 36° to simplify the problem.