Catalan Numbers: New and Old

by

Anthony Varey

A Project Submitted in
Partial Fulfillment of the
Requirements for the Degree of

Master of Science
in
Mathematics

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ABSTRACT

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Under the Supervision of Associate Professor Jeb F. Willenbring

We provide an exposition of several equivalences between interpretations of the Catalan numbers. The origins of these interpretations span almost 300 years of mathematics. A timeline for several of these interpretations is provided, beginning with a correspondence between Euler and Goldbach. The breadth and depth of the Catalan numbers is striking, as they arise in several areas, including combinatorics, complex analysis, and geometry. We survey a sizable portion of this work.

Major Professor Date

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Chapter 1

Timeline of the Catalan Numbers

1.1 Historical Summary

The Catalan numbers are a famous sequence that appears in many fields of mathematics. These areas range from geometry and combinatorics to complex analysis and algebra. Eugène Charles Catalan was credited with the discovery of the sequence of numbers, $C_n = \frac{1}{n+1}\binom{2n}{n}$ for $n = 0, 1, 2, ...$ (see [1]), and a correspondence between two interpretations.

Figure 1.1: Eugene Charles Catalan [8]

The origins of this famous sequence of numbers came 60 years prior to Catalan's birth with the work of Leonhard Euler (15 April 1707 - 18 September 1783). Euler wrote one of the first papers concerning the Catalan numbers. He was one of the very first people to ever work with the sequence when studying the triangulations of a convex n-gon. Although few details were provided, he gave a formula for the number of ways that a convex n-gon can be divided into $n-2$ triangles [1]. Euler originally gave the formula for the recurrence relation (See 1.1) around 1758, but the formula with an enumerative method would come much later [45]. Evidence of Euler's interest is provided in a 1751 correspondence with Christian Goldbach (18 March 1690 - 20 November 1764) [26].

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ten wird, auf wie vielerley verschiedene Arten solches geschehen könne. Setze ich nun die Anzahl dieser verschiedenen Arten $= x$, so habe ich per inductionem gefunden

wenn $n = 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10$

so ist $x = 1,\ 2,\ 5,\ 14,\ 42,\ 132,\ 429,\ 1430.$

Hieraus habe ich nun den Schluss gemacht, dass generaliter sey

$$x = \frac{2.6.10.14.18.22\ldots.(4n-10)}{2.3.4.5.6.7\ldots.(n-1)}$$

oder es ist $1 = \frac{2}{2}$, $2 = 1.\frac{6}{3}$, $5 = 2.\frac{10}{4}$, $14 = 5.\frac{14}{5}$, $42 = 14.\frac{18}{6}$, $132 = 42.\frac{22}{7}$; dass also aus einer jeden Zahl die folgende leicht gefunden wird. Die Induction aber, so ich gebraucht, war ziemlich mühsam, doch zweifle ich nicht, dass diese Sach nicht sollte weit leichter entwickelt werden können. Ueber die Progression der Zahlen 1, 2, 5, 14, 42, 132, etc. habe ich auch diese Eigenschaft angemerket, dass $1 + 2a + 5a^2 + 14a^3 + 42a^4 + 132a^5 + \text{etc.} = \frac{1-2a-\sqrt{(1-4a)}}{2aa}$. Also wenn $a = \frac{1}{4}$, so ist

$$1 + \frac{2}{4} + \frac{5}{4^2} + \frac{14}{4^3} + \frac{42}{4^4} + \text{etc.} = 4.$$

Euler.

Figure 1.2: Euler and Goldbach Correspondence [45]

In this correspondence, Euler provided a solution for the number of triangulations. Soon after the recurrence relation for the solution would be discovered. Johann Segner (9 October 1704 - 5 October 1777) provided a solution in the form of a recurrence relation in 1758 (see [3]), which was essentially:

$$C_{n+1} = C_1 C_n + C_2 C_{n-2} + \cdots + C_{n-2} C_2 + C_n C_1 C_1 = 1, \forall n \geq 1. \tag{1.1}$$

Segner did not give a formula for the recurrence in terms of a binomial coefficient [22], even though the binomial coefficients were discovered in the 10th century [56]. The binomial coefficients for small numbers were written down by Euclid [55]. Segner obtained his recurrence by realizing that a triangulation of a polygon can be obtained inductively from smaller polygons [3]. In 1791, Nicolas Fuss (13 Jan 1755 - 4 Jan 1826) and Johann Pfaff (22 Dec. 1755 - 21 April 1825) formulated the triangulation problem [1]. Fuss was a well known assistant to Euler, while Pfaff was famous for, among other things, the Pfaffian of a skew-symmetric matrix, as coined by Cayley [54].

The sequence, C_n, is named for Catalan, who was credited for the discovery of the connection to the number of ways to parenthesize a non-associative product of n factors, which is closely related to parenthesized expressions. His interest arose during his work on the Tower of Hanoi Puzzle [8]. Catalan did not solve Euler's polygonal division problem, but found another connection to the sequence for which he is named [59]. The paper on parenthesizing non-associative products was published in 1838 [59]. The original solution had been established in 1758, where Segner gave the recurrence relation 1.1 for the Catalan numbers [26]. The naming of the sequence of numbers for Catalan was due to his exposition [59]. The main content of the 1838 paper was on the number of distinct ways of writing pairs of parentheses [59]. The recurrence was independently discovered by a Mongolian mathematician, An-Tu Ming (1692 - 1763), in the eighteenth century [30], which would place his work prior to Catalan's discovery, and possibly Euler's as well [30]. Ming's results were not officially published until 1839, a year after Catalan's paper in 1838 [30]. There is much speculation over the dates the results were discovered, but there is no concrete evidence of Ming's discovery before Catalan [30]. It it believed that Ming was aware of this in his 1730 work *Ge Yuan Mi Lü Jie Fa*, where he used the

Catalan numbers to expand infinite series [76]. If his discovery had been made prior to Catalan's, the sequence may today be referenced as the Ming numbers [29].

The details of Euler's original work would not come until 1838. Joseph Liouville (24 March 1809 - 8 September 1882) publicized the problem of Euler's triangulation solution to obtain a more detailed solution. Gabriel Léon Jean Baptiste Lamé (22 July 1795 - 1 May 1870) answered the problem the very next day [48]. His work would later be published in the Journal de Mathématiques Pures et Appliquées [25]. This brought attention to the sequence and sparked the interest that continues today. Lamé's work inspired works by Arthur Cayley (16 August 1821 - 26 January 1895) and Joseph Louis François Bertrand (11 March 1822 - 5 April 1900). Cayley wrote papers showing that he was aware of the connection between the parenthesized expressions and plane trees [49].

The ballot problem was also considered by Bertrand in 1887, where he gave a brief description of how to obtain a solution with induction [50]. Bertrand considered this problem in 1887 when he posed the question on voting:

CALCUL DES PROBABILITÉS. — *Solution d'un problème;*
par M. **J. BERTRAND.**

« On suppose que deux candidats A et B soient soumis à un scrutin de ballottage. Le nombre des votants est μ. A obtient m suffrages et est élu, B en obtient $\mu - m$. On demande la probabilité pour que, pendant le dépouillement du scrutin, le nombre des voix de A ne cesse pas une seule fois de surpasser celles de son concurrent.

Figure 1.3: Bertrand Ballot Problem [50]

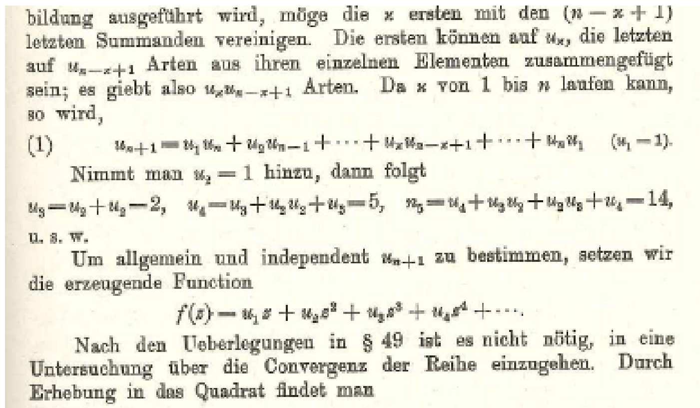
See http://webspace.ship.edu/msrenault/ballotproblem/EnglishBertrand.pdf for an English translation. Bertrand felt that such a simple result should have a direct proof method, however the solution required the invention of a new method of counting. Désiré André came up with the original counting method which we know today as André's reflection principle. André provided his detailed solution later the same year [63]. He is also credited with the creation of the principle, but never actually employed reflections in his solution [65]. Marc Renault described in his paper, *Lost (and Found) in Translation: André's Actual Method and its Application to the Generalized Ballot Problem*, how André used a modified version of the true reflection principle to provide a solution to the ballot problem [65]. For an exposition on both the ballot problem and the reflection principle, see Feller's *An Introduction to Probability Theory and its Applications* [85].

Hermann Schubert (22 May 1848 - 20 July 1911) discovered a radically different interpretation in his calculus of enumerative geometry (now called Schubert calculus) [32]. His 1879 work: *Kalkül der Abzählenden Geometrie* [68] aimed to find the number of points and lines that satisfied given geometric conditions [66]. Schubert's work came as a response to questions of enumerative geometry posed by S. Maillard [69] and Hieronymus Georg Zeuthen (15 February 1839 - 6 January 1920) [70] in 1872. At the time, there was little foundation for such works in algebraic geometry and enumerative geometry relied on intuitive counting methods [70].

Schubert calculus is the basis of one of the twenty-three famous "Hilbert Problems," which David Hilbert (23 January 1862 - 14 February 1943) delivered before the International Congress of Mathematicians at Paris in 1900 [60]. The Hilbert problem asks for a more rigorous foundation of Schubert calculus, and is the 15th Hilbert problem [60].

The name, Catalan numbers, was first used in text in 1901 in a combinatorics textbook by Eugen Netto (30 June 1848 - 13 May 1919) [2]. One of Netto's advisors was Ernst Kummer (29 January 1810 - 14 May 1893), a highly noted number theorist who discovered ideals. The book naming this sequence of numbers after Catalan was in *Lehrbuch der Kombinatorik*, and was the first book to reference them as Catalan numbers [4]. There is a striking similarity between the recurrence relations of Netto and Segner. The recurrence relation for the number of ways a convex n-gon can be divided into $n-2$ triangles was provided by Netto in [4], p. 193.

The topics of modern papers on the Catalan numbers vary from revisiting the ballot problem to folding letter clippings. The ballot problem was revisited when



bildung ausgeführt wird, möge die x ersten mit den $(n - x + 1)$ letzten Summanden vereinigen. Die ersten können auf u_x, die letzten auf u_{n-x+1} Arten aus ihren einzelnen Elementen zusammengefügt sein; es giebt also $u_x u_{n-x+1}$ Arten. Da x von 1 bis n laufen kann, so wird,

(1) $$u_{n+1} = u_1 u_n + u_2 u_{n-1} + \cdots + u_x u_{n-x+1} + \cdots + u_n u_1 \quad (u_1 = 1).$$

Nimmt man $u_2 = 1$ hinzu, dann folgt

$$u_3 = u_2 + u_2 = 2, \quad u_4 = u_3 + u_2 u_2 + u_3 = 5, \quad n_5 = u_4 + u_3 u_2 + u_2 u_3 + u_4 = 14,$$

u. s. w.

Um allgemein und independent u_{n+1} zu bestimmen, setzen wir die erzeugende Function

$$f(z) = u_1 z + u_2 z^2 + u_3 z^3 + u_4 z^4 + \cdots.$$

Nach den Ueberlegungen in § 49 ist es nicht nötig, in eine Untersuchung über die Convergenz der Reihe einzugehen. Durch Erhebung in das Quadrat findet man

Figure 1.4: Netto's Recurrence [4]

Theodore Motzkin (26 March 1908 - 15 December 1970) worked on a more general ballot problem with Aryeh (Arie) Dvoretzky(3 May 1916 - 8 May 2008) [40]. Donald E. Knuth (10 January 1938 - Present) has several exercises applying the Wallis Product Formula for π, showing how π relates to an asymptotic formula involving the Catalan numbers [72] (See exercise 1.2.11.2 - 5). He also wrote an interesting paper on the Catalan numbers entitled: "The Toilet Paper Problem" (see [28]). The sequence arises when looking at the asymptotic behavior of the average number of sheets left on two rolls of toilet paper [28]. To contrast, Roger B. Eggleton and Richard K. Guy(30 September 1916 - Present) saw the sequence appear when they considered the probability of a specific function being convex on the interval $[0, 1]$ [51]. The Catalan numbers even appeared when John A. Ewell (28 February 1928 - 21 July 2007) was researching Ramanujan's estimations of $\frac{1}{\pi}$ [52].

Arthur Stone (30 September 1916 - 6 August 2000), while drawing on strips of paper from letter clippings, discovered a figure known as a flexagon. A flexagon is a figure obtained from a strip of paper by folding and pinching, but without tearing or cutting. He later realized with the help of Richard Feynman (11 May 1918 - 15 February 1988), Bryant Tuckerman (28 November 1915 - 19 May 2002), and John W. Tukey (16 June 1915 - 26 July 2000) that the number of ways of folding a flexagon with a known number of pats[1] is exactly the Catalan sequence [42]. Wolfdieter Lang also wrote a paper on derivatives of polynomials with a connection to the generating function for C_n [53]. There is also great research involving determinants and Catalan numbers. One great source for this connection is Christian Krattenthaler's *Determinants of (Generalised) Catalan Numbers*, which contains references to many other authors' work relating determinants to Catalan numbers. [27]. In 1995, Adriano Garsia and Mark Haiman [24] introduced a set of conjectures known as diagonal harmonics. These conjectures were later verified in 2002 in later work titled *Vanishing theorems and character formulas for the Hilbert scheme of points in the plane* (see [5]) which were also proved by Haiman.

The goal of this project is to sample the highlights of the Catalan numbers through time, and provide a bijective correspondence between each distinct interpretation sequentially. The subsequent section provides a timeline to highlight significant points of interest.

1.2 Timeline

This section provides a pictorial view of the history of the Catalan numbers, as outlined in section 1.1. The timeline provides a compact description of when discoveries were made, and also when they were made relative to the lives of their discoverers. Each section is organized in a chronological fashion. Chapter 2 contains additional historical detail of each interpretation, and definitions to introduce each in a modern context. Chapter 3 contains the interpretations. It should be noted that this is only a small portion of the full context of interpretations of the Catalan numbers. For a list of hundreds of interpretations, one should consult Richard P. Stanley's combinatorics text (see [1]) for a sizable compilation of interpretations, although the project presented here is not subsumed in Stanley's exposition.

[1] A **pat** is a polygon used to make flexagons, sometimes called leaves [61].

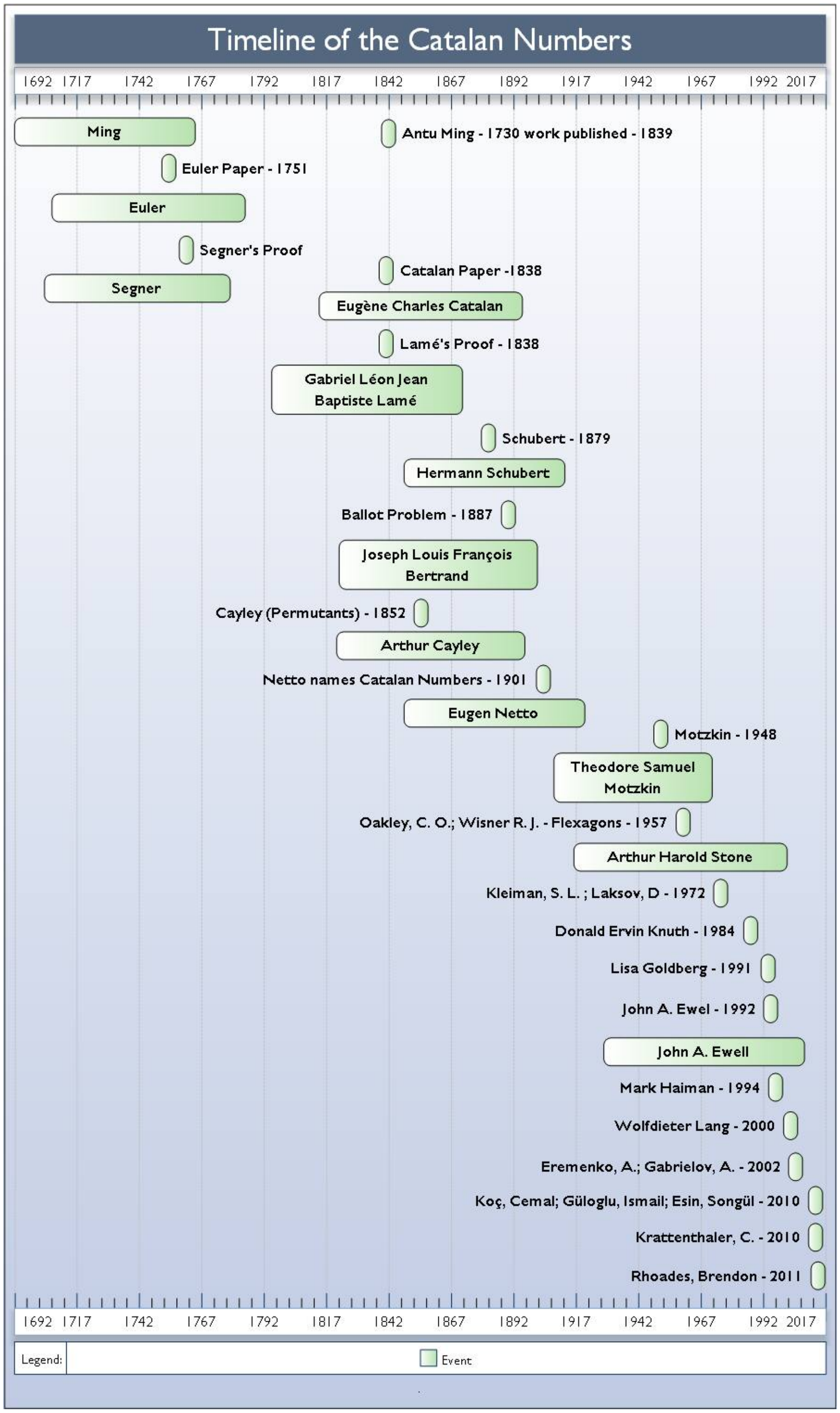


Figure 1.5: A Timeline of the Catalan Numbers

Chapter 2

Interpretations Through Time

In this chapter, we introduce several distinct interpretations of the Catalan numbers organized chronologically. We begin each section with precise definitions to place the interpretation into a modern context. Then, provide the historical impact of the particular interpretation. Some terminology will be introduced in a section, although used in a later section.

2.1 Generating Function

The formal manipulation of power series dates back to the eighteenth century. In 1728, the generating function was introduced in order to find an explicit formula for the Fibonacci numbers [11]. Many of these same techniques and principles are still used today in combinatorics, probability, and algebra for a variety of sequences [43], page *xiv*. In this section, we introduce some definitions used to describe generating functions as they relate to the Catalan numbers. Then, take a look at how these evolved over time to become critical to many fields of study.

2.1.1 Definitions

Many combinatorial counting problems with a solution, a_n, depending on a parameter n, can be solved by using the recursion relation for the sequence to find an explicit formula for the sequence [62]. We introduce the notion of a formal power series for this sequence.

A **formal power series** over a field F is an expression

$$\sum_{n\geq 0} a_n t^n = a_0 + a_1 t + a_2 t^2 + \ldots$$

, where $n \in \mathbb{N}$ (See [31]), page 54. More formally, a power series is an infinite sequence $(a_0, a_1, a_2, \ldots)$ of elements of F [31], page 54. The **Generating Function** is a power series whose coefficients form the number sequence in question [31], page 49.

2.1.2 History

The origins of the generating function date back to 1728 by Daniel Bernoulli, and independently to de Moivre in 1730 [11]. Both used the idea of a generating function to find an explicit formula for the Fibonacci numbers [11].

Euler introduced the idea of a certain type of polynomial in order to evaluate the alternating zeta function [84]. He gave examples of how he applied these functions in many of his works including *Institutiones calculi differentialis* in 1755 [86]. Several translations of these texts are available, see [46] for references. Some of these methods led to the development of the idea of what we know of today as a generating function.

2.2 Triangulations of a Convex Polygon

In the years 1751 to 1858, the Catalan numbers were interpreted as a count of "triangulations of a convex polygon." We review this history after precisely stating the relevant terminology in the context of graph theory. Graphs were first studied by Leonhard Euler in his study of the Königsberg bridge problem [78].

2.2.1 Definitions

We recall some terminology with the intent of defining a triangulation of a polygon.

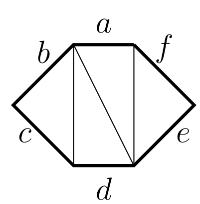
A **graph** G consists of a set V (or V(G)) of **vertices**, a set E (or E(G)) of **edges**, and a mapping associating to each edge $e \in E(G)$ an unordered pair xy of vertices (x and y) called the **endpoints** (or simply the **ends**) of e [62], page 1. An edge is **incident** with its ends, and it **joins** the ends. We allow $x = y$, x, $y \in V$, in which case the edge is called a **loop**. [62], page 1. We may represent a graph by a **drawing** where we represent each vertex by a point in the plane, and the edges by line segments joining some of the pairs of points [62], page 1. The graph is called **planar** if it can be drawn in the plane such that no two edges(that is, the segments representing the edges) cross [62], page 1. We call a graph **simple** when it has no loops and no two distinct edges have exactly the same pair of ends [62], page 2. A graph is **finite** if the sets $E(G)$ and $V(G)$ are both finite sets [62], page 4.

The number of edges incident with a vertex x is called the **degree** or **valency** of x [62], page 4. Two vertices x and y of a graph G are called **adjacent** if they are distinct and joined by an edge. [62], page 4. A **walk of length k** in a graph G is a succession of k edges of G of the form: uv, vw, wx, . . . , yz. [74], page 34.

If all the edges (but not necessarily all the vertices) of a walk are different, then the walk is called a **trail**. If, in addition, all the vertices are different, then the trail is called a **path** [74], page 35. A **closed walk** in a graph G is a succession of edges of G of the form: uv, vw, wx, . . . , yz, zu. If all of these edges are different, then the walk is called a **closed trail**. Assuming, in addition, the vertices u,v,w,x, . . . ,z are all different, then the trail is called a **cycle** [74], page 35. If a path from x to y exists for every pair of vertices x, y of G, then G is called **connected**. Whenever this is not the case, then G consists of a number of connected **components**. We say that the null graph with no vertices and no edges is not connected [62]. A common convention is to denote the path from x to y as an $x - y$-path.

A **planar drawing** of a simple graph G partitions the complement of the drawing into open subsets of the plane. We will call the bounded subsets **faces**. The **degree** of a face, F, is the number of edges on the boundary of F. We call a face with degree three a **triangular face**. A face, F, is said to be convex if for any points p and q in F, the line segment from p to q is contained in F. A **polygon** is a cyclic graph, P, with at least three vertices, together with a planar drawing with a single convex face, called the **interior**. If an edge, e, is added to two non-adjacent vertices of a polygon and drawn in the interior, then we say that e is a **diagonal**. A **triangulation** of a polygon, P, is a graph obtained from P by adding diagonals to P such that the resulting graph is planar and has triangular faces.

Our main focus is on counting the number of triangulations of a convex polygon. There are several ways to triangulate a given convex polygon. Here is an example of one way to triangulate a hexagon:



2.2.2 History

Polygons as mathematical objects have been used since the Greek period ($\sim$ 8th century BCE - 6th century CE), and there are hints of their use dating back as far as the Egyptians (3150 BCE - 332 BCE) and Babylonians ($\sim$ 2000 - 600 BCE) [73]. There is evidence of Egyptians using them in architecture in the pyramids, as well as theoretically to find the slope to build mastabas (tombs) [73]. The Egyptian mathematics was not solely based on primitive architecture. There is evidence, four problems on the Rhind Mathematical Papyrus, of a student or teacher solving basic algebraic equations as well [20]. The Babylonians, in comparison, were able to carry out complex calculations such as square roots and find Pythagorean triples [21].

The triangulations of a convex polygon is regarded as the first interpretation of the Catalan numbers [45]. Euler's discovery in 1751 came before they were referred to as the Catalan numbers, and is one of the main interpretations people associate with the Catalan numbers. He did not provide an rigorous proof of the triangulation problem, but gave a formula for the number of triangulations of a convex polygon [45]. The solution of the triangulation problem is credited to two people, Segner and Lamé.

Figure 2.1: Johann Andreas Segner [18]

Johann Adreas Segner was born Oct 9, 1704 in what is now known as Bratislava, Slovakia [18]. He studied both mathematics and medicine at Pozsony's Lyceum in Hungary. Later, he moved on to be a professor at Halle with assistance from Euler in 1755 [18]. Shortly after his move to Halle, in 1758, Segner was the first to provide a proven counting method and a recurrence relation for the number of triangulations [22]. A descriptive method of counting the triangulations of a polygon would come much later with the work of Gabriel Lamé (22 July, 1795 - 1 May, 1870).

Figure 2.2: Gabriel Lamé [19]

To contrast, Lamé's method counted the number of total triangulations of a convex polygon, and remove the number of repeated triangulations [25]. Though both Segner and Lamé both solved the triangulation problem, Netto's combinatorics textbook is the source of reference naming the sequence for Catalan [4].

2.3 Parenthesized Expressions

Catalan's famous paper on non-associative products is closely related to parenthesized expressions, and is the foundation from which this famous sequence draws its name. First, we will define a parenthesized expression. Then, we will retrace the history as it pertains to this particular interpretation.

2.3.1 Definition

Here, we introduce the idea of a parenthesized expression inductively.

A **parenthesized expression** of length $2n$ is a finite sequence of elements "(" and ")" of the form:

a. () if $n = 1$

b. $(\alpha)\beta$ where α and β are parenthesized expressions of lengths adding up to $2(n-1)$.

We consider the two expressions (())() and ()(()) to be different. The following table lists all of the distinct parenthesized expressions for $n = 1, 2, 3, 4$:

n	List of Parenthesized Expressions	Count
1	()	1
2	()(), (())	2
3	()()(), (())() , ()(()) , ((())) , (()())	5
4	(((()))), ((()())), ((())()), ((()))(), (()(())), (()()()), (())(()), (())(()), (())()(), ()((())), ()(()()), ()(())(), ()(()()), ()()()()	14

Figure 2.3: List of Parenthesized Expressions

2.3.2 History

The sequence of the Catalan numbers is named by Netto for Catalan's discovery of the connection of the sequence to the placing of parenthesis around non-associative products, which is related to the parenthesized expressions and polygonal triangulation problem [4].

Eugene Catalan's 1838 paper [59] counted the number of ways of placing parenthesis around a non-associative product of elements. This problem has a close relation to rearranging pairs of parentheses. Cayley also briefly worked on parenthesized expression problem as it pertains to plane trees [49]. The parenthesized expressions have been studied by a mathematical physicist, Ernst Schröder (25 November 1841 - 16 June 1902), who was interested in the problem [16].

2.4 Binary Trees

In this section, we introduce graph theoretic terminology to describe the notion of a rooted binary tree. Then, discuss the history as it pertains to the Catalan numbers and trees. The key mathematician who interpreted the Catalan numbers using the notion of a binary tree was Arthur Cayley.

2.4.1 Definitions

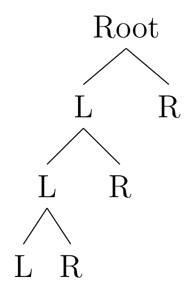
Here, we introduce modern terminology with the intention of describing rooted binary trees.

A **subgraph** of a graph G is a graph H such that $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$ and the ends of an edge $e \in E(H)$ are the same as its ends in G. We call the subgraph H of a graph G a spanning subgraph when $V(H) = V(G)$ [62], page 4. If a connected graph contains no cycles, we call it a tree **tree** [62], page 6. A **directed graph** or **digraph** D is a finite nonempty set of objects called vertices together with a (possibly empty) set of ordered pairs of distinct vertices of D called arcs or **directed edges** [79], page 14. A digraph D is called an **asymmetric digraph** or an **oriented graph** if whenever uv is an arc of D, then vu is not an arc of D [79].

A **directed tree** is an asymmetric digraph whose graph is a tree [79], page 77. A **rooted tree** is a directed tree T with some vertex r (called the **root**) such that T contains a $r, \dots, v$ path for every vertex v of T [79], page 77. In a rooted tree with a root vertex x, the vertices which lie on the path from x to a different vertex v are called the **ancestors** of v [71], page 192. Given a rooted tree, for any vertex

$v \neq r$, the **father** of v is that unique vertex u that is adjacent to v. Conversely, v is the **son** of u. Two vertices having he same father are **brothers** [79], page 78.

The vertices of a rooted tree having no sons are called **leaves**. All other vertices (those with sons) are called **internal vertices** [79], page 78. If every vertex of T has either n or no sons, then T is a **complete n-ary tree** [79], page 78. Instead of referring to the 2-ary tree as such, we often call it a binary tree. A rooted tree T is **ordered** if the sons of each vertex of T are ordered (as first son, second son, and so on). In a drawing of an ordered tree, the sons are ordered from left to right [79]. As an example, the following is a rooted binary tree with seven vertices:



2.4.2 History

Trees have varying applications from combinatorics to network theory and computer science. More recently, they have gained momentum as a result of the computer age. This had led to several applications varying from Internet searches to artificial intelligence.

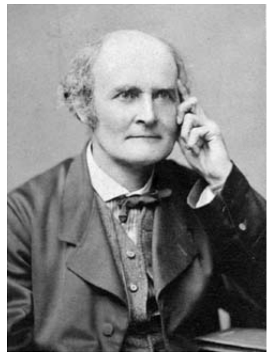


Figure 2.4: Arthur Cayley [17]

Arthur Cayley showed that he was aware of a connection to the Catalan numbers using the binary trees [49]. He showed that he was aware of a connection between the parenthesized expressions and binary trees [49]. There is a close relation between the binary trees Cayley studied and the full rooted binary trees discussed in this section [49].

2.5 Monotonic Paths

There are three separate interpretations involving paths that are interrelated. Monotonic paths have a close relation to rooted binary trees. A Dyck path is another studied in combinatorial theory. An important problem connected to both is Bertrand's famous Ballot Problem. We introduce some terminology to describe each of these ideas, then survey a portion of their historical significance.

2.5.1 Definitions

The following is terminology needed to introduce monotonic paths with the aim of describing and counting the number of paths, both monotonic paths and Dyck paths.

The set $[n] \times [n]$ is called the **grid** of size n, where $n \in \mathbb{Z}^+$ and $[n] = \{1, 2, \ldots, n\}$. A **monotonic path** on an $n \times n$ grid is a finite sequence of steps from $(0, 0)$ to (n, n) with steps $(x, y) \to (x, y + 1)$ or $(x, y) \to (x + 1, y)$, where x,y $\in [n]$. Paths

P in the (x,y) plane from $(0, 0)$ to $(2n, 0)$, with steps $(1, 1)$ and $(1, -1)$ that never pass below the x-axis are called **Dyck Paths** [1].

The monotonic paths have a variety of uses. One such application of the monotonic paths was the ballot problem, where each path is used to represent votes for a particular candidate. The Ballot Problem (as posed by Bertrand): Suppose that candidates A and B are running in an election. If a votes are cast for A and b votes are cast for B, where $a > b$, then the probability that A stays ahead of B throughout the counting of the ballots is $\frac{(ab)}{(a+b)}$ [50].

2.5.2 History

The real world application of monotonic paths aim to answer questions ranging from postal routes and wiring computer processors, to plowing snow and flight planning [31]. Feller also used Dyck Paths along with the reflection principle to prove Bertrand's Ballot Problem [85].

Figure 2.5: Joseph Bertrand [10]

The ballot problem was initially posed by Joseph Bertrand in 1887 [50] in response to other works he was studying [77]. Bertrand was a French mathematician who both lived and died in Paris from his birth in March 11, 1822 to his end April 5, 1900 [47]. He mentioned that he had studied works by such mathematicians as Huygens, de Moivre, Laplace, Lagrange, and Ampere with the intention to find a solution to the ballot problem [77].

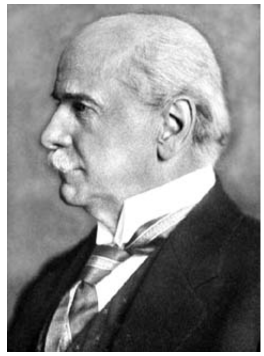


Figure 2.6: Walther Franz Anton von Dyck [13]

Walther Franz Anton von Dyck (6 December 1856 - 5 November 1934) was a german mathematician from Munich credited as being the first person to defined a mathematical group in the modern sense following the influential work of Évariste Galois [11]. He also invented the Dyck language which consists of balanced pairs of parenthesis, much like the parenthesized expressions [11]. The Dyck words also have a close relation to the ballot problem studied by Bertrand, and were used by Feller in a solution to the ballot problem [85].

Though several proofs of the ballot problem exist, the most notable of the solutions is the reflection principle. This problem inspired the work of Désiré André [63] and the reflection method to which he is attributed. Though he did not directly employ the reflection method in his solution, his idea was later refined into the reflection principle [65], page 1. André's method is essentially Kelvin's method of images placed in the context of enumerative combinatorics [83]. One of the first people to reference the reflection method to André was J. L. Doob 1953 [81]. Other subsequent work on a generalized ballot problem was done by Dvoretzky and Motzkin [40] [64].

2.6 The Reflection Principle

In this section, we examine the reflection principle and its connection to monotonic paths and the ballot problem. First, we introduce some modern language in order to place the reflection principle in modern context. Then, survey the landscape of the reflection principle's history.

2.6.1 Definitions

The Reflection Principle is stated in Feller's probability text as follows:

Proposition 2.6.1 ([85]**, page 72)** *The number of paths from A to B which touch or cross the x-axis equals the number of paths from A' to B.*

See in Chapter 3 for an accompanying diagram.

2.6.2 History

The study of the ballot problem in 1887 [50] led to the discovery of the reflection principle [63]. Joseph Louis François Bertrand obtained his inspiration to study the ballot problem from Huygens, de Moivre, Laplace, Lagrange, and Ampere [77]. Though he did not rigorously prove his solution, he mentioned in his solution that he felt there was a simple way to obtain a solution with induction [50]. Shortly after his paper was published, André published a response providing a solution.

Désiré André's method for solving Bertrand's ballot problem led to the development of the reflection principle [63]. While André did not directly employ this principle himself, his method led to its refinement and wide use, as discussed in section 1.1 [65]. He was the first to solve the ballot problem using the method of subtracting the number of bad ballot permutations from the number of all possible ballot permutations [65]. The modern day reflection method is a variation of André's method due to Aebly and Mirimanoff [65]. Feller's probability text has a great exposition on the reflection principle [85].

2.7 Young Diagrams

In this section we introduce Young tableaux and give a correspondence between a collection of standard young tableaux and monotonic paths.

2.7.1 Definitions

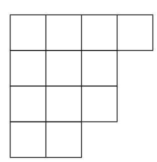
We introduce the Young diagrams and tableaux. A **partition** λ of a positive integer n is a sequence $(\lambda_1, \ldots, \lambda_k) \in \mathbb{Z}_+^k$ satisfying $\lambda_1 \geq \cdots \geq \lambda_k$ and $\sum_{i=1}^k \lambda_i = n$. See [1], page 1. The **Young Diagram** is a collection of boxes, or cells, arranged in left-justified rows, with a (weakly) decreasing number of boxes in each row [82], page 1. A **Young tableau**, or simply a **tableau**, is a filling that is:

1. weakly increasing across each row.
2. strictly increasing down each column.

See [82], page 2

A **standard tableau** is a tableau in which the entries are the numbers from 1 to n, each occurring once [82], page 2. We say that the tableau is a tableau **on** the diagram λ, or that λ is the **shape** of the tableau [82], page 2.

As an example, note we can draw the Young diagram with the shape $(4, 3, 3, 2)$:



Observe the following fact that immediately follows from the restriction that the numbers be increasing in both the rows and columns.

Lemma 1 *The upper left hand corner of a standard Young Tableau always contains a* 1*.*

The lemma follows from the fact that the standard Young tableaux must be increasing in both the rows and columns.

Now, we introduce the standard Young tableaux and its connection to the Catalan numbers. We exhibit the case where $n = 3$, there are 5 such standard Young Tableau with shape $(3, 3)$:

$$\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5&6\\\hline\end{array}, \quad \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5&6\\\hline\end{array}, \quad \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4&6\\\hline\end{array}, \quad \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5&6\\\hline\end{array}, \quad \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4&6\\\hline\end{array}.$$

Figure 2.7: Standard Young Tableau with Shape $(3, 3)$

The fact that there are five such standard Young Tableau with the shape $(3, 3)$ has the following generalization:

Proposition 2.7.1 *There are C_n standard Young Tableau with shape (n, n).*

Proof See [31]. $\blacksquare$

2.7.2 History

A Young Tableau is an object that has uses when studying partitions, combinatorics, representation theory, and the Schubert calculus of the Grassmannian [82]. These diagrams are closely related to the Ferrers diagram of a partition. The tableaux were originally discovered in 1900 by Alfred Young. He was a British mathematician born in Widnes, Lancashire, England April 16, 1873 and died December 15 , 1940 [67]. In 1901, he was appointed to be a lecturer at Selwyn College in Cambridge, and ended up transferring to Clare College in 1905. In 1908, Young became a clergyman, and then in 1910 he became a parish priest in Essex. He returned to lecturing at Cambridge in 1926.

It was from the invention of the tableaux where people such as Frobenius, Hodge, Rota, Stanley, and Gessel furthered the study of the symmetric group and other algebraic objects. Frobenius is well known for his theory of group representations and characters. William Vallance Douglas Hodge (17 June 1903 - 7 July 1975) is most notably known for the famous, unsolved problem in algebraic geometry, the Hodge Conjecture. Gian-Carlo Rota (27 April, 1932 - 18 April, 1999) popularized the study of incidence algebras among the combinatorialists, and worked on several other fundamental probability problems. He was also the advisor of Richard Stanley. Stanley is known for his standard textbook in combinatorics, *Enumerative Combinatorics*, and the many interpretations of the Catalan numbers it contains [1]. Ira Gessel, a student of Stanley's, has written several combinatorial papers on paths and the hook length formulae [9]. More recently, Peter Striegel's capstone project, *Standard Young Tableaux and Reflected Random Walks*, gave and equivalence between the random paths with the Young tableaux [23]. This joint work with R. Stockbridge and J. Willenbring sought to find the probability of a path from the origin to a point below the diagonal to be a "good" path using the hook length formula and the reflection principle [23].

2.8 Schubert Calculus for the Grassmannian of 2-Dimensional Subspaces in $\mathbb{C}^n$

In 1900, David Hilbert provided a list of 23 open questions for the twentieth century [34], page 1. The 15th of these is to provide a rigorous foundation of Schubert's calculus of enumerative geometry [34], page 3. In this section, we introduce some terminology to find the number of projective subspaces of codimension 2 that intersect $2n$ lines in general position in projective space $\mathbb{P}^{n+1}$, and find that the number of lines incident is counted by the Catalan number, C_n [38], page 1. Then, review the background associated with the Schubert calculus.

2.8.1 Definitions

In order to describe these geometric and enumerative ideas, we must first introduce modern language to define affine and projective space, and related geometry. **Affine n-space** is the space of n-tuples $(a(1), a(2), \ldots, a(n))$ of complex numbers generally written A^n [66], page 3.

A point P of **projective n-space** (P^n) is defined by an $(n+1)$-tuple $(p(0), p(1), \ldots, p(n))$ of complex numbers not all are zero, modulo multiplication by a complex scalar. The

$p(i)$ are called the **coordinates** of P. Another $(n+1)$-tuple $(q(0), q(1), \ldots, q(n))$ also defines P if and only if there is a number c satisfying $p(i) = cq(i)$ for $i = 0, \ldots, n$. See [66], page 3. A **linear space** L in P^n is a set of points $P = (p(0), p(1), \ldots, p(n))$, where each coordinate $p(i)$ satify a system of linear equations $\sum_{i=0}^{k} a_{\alpha j} p(j) = 0$, and $\alpha = 1, 2, \ldots, (n-d)$ [66], page 3.

We say that L is **d-dimensional** if these $(n-d)$ equations are independent, that is if the $(n-d) \times (n+1)$ matrix of coefficients $[b_{\alpha j}]$ has a non-zero $(n-d) \times (n-d)$ minor [66], page 3. As a note, if the $(n-d) \times (n+1)$ matrix of coefficients $[b_{\alpha j}]$ has a non-zero $(n-d) \times (n-d)$ minor, then by linear algebra there must be $d+1$ points $P_i = (p_i(0), \ldots, p_i(n)) \in L$ with $i = 0, \ldots, d$ which span L [66], page 4. We call L a **line** if $d = 1$, a **plane** if $d = 2$, and a hyperplane if $d = (n-1)$. We also call a d-dimensional linear space a **d-plane** for short [66], page 4.

A function p on the set of all sequences $j_0 \ldots j_d$ with $0 \leqq j_\beta \leqq n$, $\beta = 0, \ldots, d-1$, which satisfies the following formulas is called an **alternating function**:

a. $p(j_0 \ldots j_d) = 0$ if any of the two j_β are equal.

b. $p(j_0 \ldots j_d) = -p(j_0 \ldots j_{\beta-1} j_{\beta+1} j_\beta j_{\beta+2} \ldots j_d)$

See [66], page 4.

The function p above may be viewed as a coordinate in a Grassmann algebra.

2.8.2 History

In 1874, Hermann Csar Hannibal Schubert (22 May 1848 - 20 July 1911) published his paper on algebraic geometry that dealt with enumeration problems of geometric objects satisfying some prescribed conditions [32]. His paper was based on what was already known in the area, but most of the ideas at the time were either on the weak foundation of the time, or an intuitive method of counting [70]. Schubert was one of the leading developers in algebraic geometry [12]. He won a gold medal from the Royal Danish Academy of Sciences for solving a problem posed by Zeuthen on the extension of the theory of characteristics in cubic space curves [12]. For solving this problem, Schubert calculus was named after him [12].

The aim of Schubert and mathematicians of the late 1800's was to build a better

Figure 2.8: Hermann Csar Hannibal Schubert [12]

foundation for algebraic geometry using topology and topological counting methods [70]. The goal of the Schubert Calculus was to answer enumerative geometric problems using algebraic concepts [32]. They wanted to be able to determine the number of linear subspaces that satisfy certain incidence conditions [32]. These types of questions arise in many areas of algebraic geometry including intersection theory.

One such interesting enumerative problem in the Schubert calculus asks: How many lines in $\mathbb{CP}^3$ intersect four fixed generic lines? It turns out, the solution to this problem is that there are only two such lines.

2.9 Branched Coverings

The idea of a branch set arises when studying the fixed critical points of a rational map in complex analysis. There are $\rho(d)$ conjugacy classes of degree d monic polynomials in one complex variable, all of whose critical points are fixed [39], page 4. In order to describe these sets, we must first introduce some terminology used when describing a branched covering. Then, discuss the historical background to better understand why such ideas were generated.

2.9.1 Definitions

The main points of interest of this section are the $PGL(2, \mathbb{C})$ orbits of rational functions. A rational map of $\mathbb{CP}^1$, the Riemann Sphere, is non-constant is an analytic, branched covering of $\mathbb{P}^1$ by $\mathbb{P}^1$ [39], page 1.

A nonconstant **rational map** can be expressed as a quotient of polynomials $R(z) = \frac{P(z)}{Q(z)}$, where $P, Q \not\equiv 0$, and P, Q not both constant [39], page 1. The **degree** of R is the number of preimages of any regular value; equivalently, degree $R = max\{degreeP, degreeQ\}$ [39], page 1. A point $c \in P^1$ is a **branch point** of R if and only if it is a critical point (the derivative is zero), and from the Riemann-Hurwitz formula, it follows that the (not necessarily distinct) critical points of a degree d map number $(2d-2)$. See [39], page 1.

If $B = \{c_1, c_2, \ldots, c_{d-1}, \infty, \infty, \ldots, \infty\}, c_i \in \mathbb{C}$ for $i = 0, \ldots, d-1$, then B is the **branch set** of any map of the form $K_1 \int \prod_1^{d-1}(z - c_i) + K_2$, where K_1, $K_2 \in \mathbb{C}$. See [39], page 1. The **multiplicity** of $c_i \in B$ is the number of j's for which $c_i = c_j$. B is **admissible** if no c_i has multiplicity greater than $d-1$, where d is the degree of the rational map [39], page 2. Two rational functions f_1 and f_2 will be called **equivalent** if $f_1 = f \circ f_2$, where f is a fractional-linear transformation. Equivalent rational functions have the same critical sets. This gives rise so the following lemma:

Lemma 2 *The branch set of a rational map is admissible* [39]*, page 2.*

Proof See Goldberg [39], page 2. ∎

2.9.2 History

The roots of Algebraic Geometry date back as far as the Hellenistic Greeks, and problems such as the Delian problem, from the 5th century BCE [15]. There are references of Arabic mathematicians, in the 10th century CE by Ibn al-Haytham, being able to solve cubic equations by purely algebraic means and interpret them geometrically [14].

The true start of algebraic geometry comes from the invention of "analytic geometry" by Fermat and Descartes [15], page 3. The main point proved during this period shows that all curves of degree 2 are conics, as proved by Fermat [15], page 4. The works of Arthur Cayley, Edward Laguerre, and Plücker sought to bring algebraic ideas into a geometric setting [15].

In 1851, when Bernhard Riemann studied algebraic geometry following the advent of the theory of a complex variable by Cauchy, Riemann used some of the ideas to attack the problem of abelian integrals, and the extension of Cauchy's theory to a theory of "surfaces [15]." These ideas led to the works of Dedekind and Zariski and their notions of ideals and topology [15]. In the 1950's, André Weil observed that the Zariski topology can be defined on abstract varieties, making describing these ideas much easier with the use of geometric terminology [15].

The ideas from this field of study were fundamental to the proof of Fermat's Last Theorem [80], page 54. The branched sets of a non-constant meromorphic function arise in Belyĭ's theorem, which is applied in the proof of the Taniyama-Shiumra conjecture [80], page 54. This interpretation also has a close connection to the Schubert Calculus [32].

2.10 Diagonal Harmonics of the Symmetric Group

The conjectures commonly called diagonal harmonics were introduced in 1995 by Adriano Garsia and Mark Haiman [24]. First, we introduce the terminology associated to this interpretation. Following the terminology, we discuss the significance of these conjectures.

2.10.1 Definitions

Here, we introduce modern language needed to describe the conjectures referred to as diagonal harmonics. To a bigraded S_n module $A = (A)_{0,0} \oplus (A)_{0,1} \oplus (A)_{1,0} \oplus \dots$, we attach a Frobenius Series: $\mathfrak{F}_A(t,q) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} t^i q^j \phi((A)_{i,j})$ (see [24], page 7). If we have bigraded spaces $A = (A)_{0,0} \oplus (A)_{0,1} \oplus (A)_{1,0} \oplus \dots$, we write a bivariate Hilbert series such as: $\mathfrak{H}_A(t,q) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} t^i q^j dim((A)_{i,j})$ (see [24], page 7).

2.10.2 History

The Laplace operator of f, $\nabla^2 f$ (see [35]), is named for Pierre Simon de Laplace, who initially used this idea in the study of celestial mechanics of the solar system [36]. He also showed that the potential function satisfied the equation $\nabla^2 f = 0$ (Laplace's equation) [37]. In complex variables, it is a well known fact that a function u is harmonic if it has continuous second partial derivatives and $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ (Laplace's equation) [75], page 252.

The angular portion of a set of solutions to Laplace's equation is known as spherical harmonics [36]. Laplace first introduced the idea that a specific set of spherical harmonics forms an orthogonal system [36]. The Laplace Operator, over time, has been generalized and applied in fields ranging from theoretical physics to Lie theory.

In 1995, Adriano Garsia and Mark Haiman published a list of conjectures referred to as "diagonal harmonics" [24]. They consider a ring, R/I, where R is a polynomial ring in two sets of variables $\{x_1, \cdot, x_n\}$, $\{y_1, \cdots, y_n\}$, and I is the ideal generated by the non-constant invariants under the diagonal action of the symmetric group [24]. The quotient is therefore a representation of the symmetric group [24]. They conjecture (and Haiman later proves) that R/I has dimension $(n+1)^{(n-1)}$, which is the number of labeled trees on n+1 vertices (Cayley's theorem). The isotypic component of the sign representation has dimension equal to the Catalan number [24].

Chapter 3

Interpretations with Examples

There are a multitude of interpretations of the Catalan numbers in varying contexts of mathematics. We view through a historical and chronological lens the interpretations over nearly 300 years of mathematics (see Chapter 1 1.5). This begins with the original works by Euler and Segner [45], and ends with recent discoveries by Goldberg [39].

3.1 Generating Function

Since the eighteenth century the notion of a formal power series has been applied in many fields of mathematics. This method has also been adapted over time for many sequences, like the Fibonacci numbers and the Catalan numbers to name a few. The generating function for the Fibonacci numbers is $g(x) = \sum_{m=0}^{\infty} F_m x^m$, where $m \in \mathbb{Z}_{\geq 0}$ and F_m is the m-th Fibonacci number. Similarly, for the Catalan numbers the series is $f(x) = \sum_{n=0}^{\infty} C_n x^n$, where $\mathbb{Z}_{\geq 0}$ and C_n is the n-th Catalan number. In this section we provide the generating function, as well as another recurrence relation for the Catalan numbers.

It is convenient to have different formulaic interpretations of the sequence for later use:

Proposition 3.1.1 $\sum_{n=0}^{\infty} C_n x^n = \frac{1-\sqrt{1-4x}}{2x}$

Proof See Frazer Jarvis's work [6]. $\blacksquare$

The generating function naturally gives rise to the recursion relation:

Proposition 3.1.2 $C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}$ *for* $n \geq 0$ *is a recurrence relation which returns the Catalan numbers where* $C_0 = 1$*.*

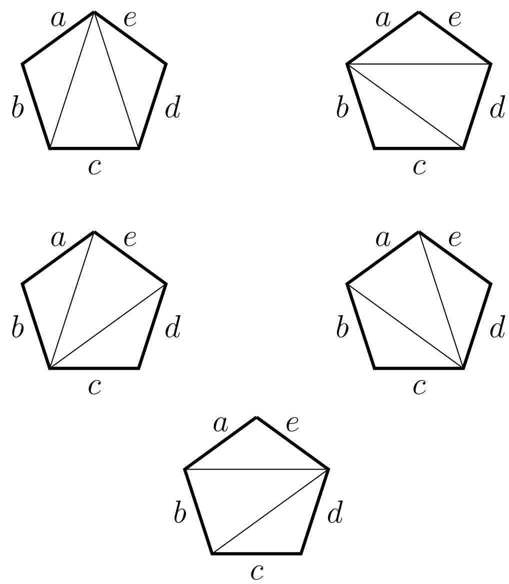
Proof See Jarvis [6]. $\blacksquare$

These two propositions will help in counting triangulations, parenthesized expressions, and other interpretations.

3.2 Triangulations of a Convex Polygon

The triangulations of a convex polygon were of interest to people such as Euler, Segner, Lamé, and Catalan. They were interested in the number of ways that a convex polygon can be dissected into triangles. A natural question arises: how many triangulations of an convex polygon are there?

We generate the triangulations for $n = 5$, and note that there are 5 ways to triangulate a pentagon.



The fact that there are 5 such triangulations of a pentagon motivates the following.

Proposition 3.2.1 *C_n is the cardinality of the set of triangulations of convex polygons.*

Proof Several expositions exist. See [22] and [25]. For a more modern solution in the context of combinatorics, see [44]. ∎

In principle, we think of a diagonal of a convex polygon splitting it into two smaller convex polygons. Given a polygon with n sides, and the diagonal splits it into a convex polygon with j sides, and a polygon with $n-j+2$ sides. The following image illustrates:

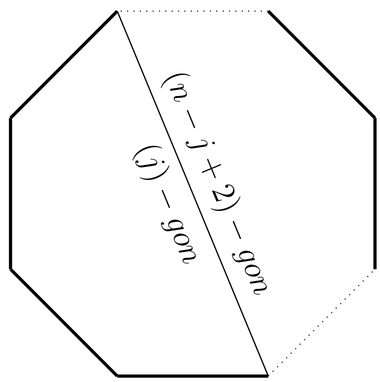


Figure 3.1: Triangulation of a Convex Polygon

This method is similar to the counting method used in counting the number of rearrangements. One can see that the recurrence relation 1.1 for the Catalan numbers pops out almost immediately in counting the number of triangulations of the smaller convex polygons.

3.3 Parethesized Expressions

The parenthesized expressions are an area of interest closely related to the non-associative multiplication problem considered by Eugene Catalan. In his problem, he considered the number of ways of multiplying a set of elements together two at a time.

Here, we generate a list of distinct parenthesized expressions of length $2n$ for a fixed n:

n	List of Parenthesized Expressions	Count
1	()	1
2	()(), (())	2
3	()()(), (())() , ()(()) , ((())) , (()())	5
4	(((()))), ((()())), ((())()), ((()))(), (()(())), (()()()), (())(()), (())(()), (())()(), ()((())), ()(()()), ()(())(), ()(()()), ()()()()	14

Figure 3.2: List of Parenthesized Expressions

From this list, we can see the number of ways the parentheses can be arranged should be counted by the Catalan numbers. This idea leads us to the following proposition:

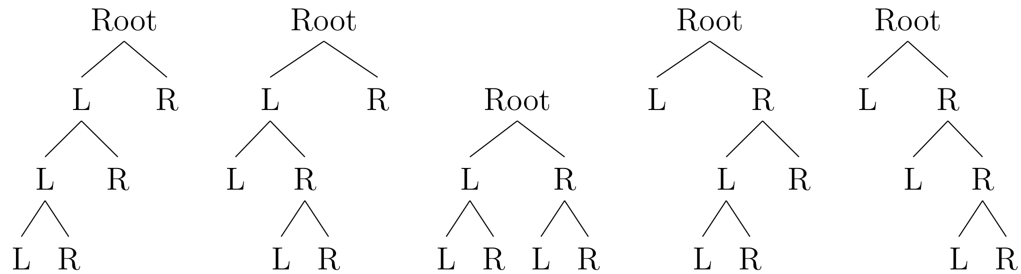
Proposition 3.3.1 *There is a bijective correspondence between the set of triangulations of a convex polygon and the set of parenthesized expressions.*

Proof See Catalan's *Note sur une équation aux différences finies* [59], or for a combinatorial exposition, Gross's text gives a modern approach [57]. ∎

If we are given a parenthesized expression $(\alpha)\beta$, there is a first time in which the number of left parentheses equals the number of right parentheses. Similar to counting the number of rearrangements of the triangulations, one can count these expressions inductively.

3.4 Binary Trees

The binary trees, and their connection to parenthesized expressions, were of interest to Arthur Cayley [49]. In specific, the trees we are interested in are the full binary trees. Every internal node of the full rooted binary tree has exactly two children. We enumerate the case of $n = 3$, which means the tree has $2(3) + 1 = 7$ internal nodes:

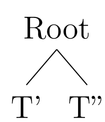


The fact that there are five such rooted binary trees motivates:

Proposition 3.4.1 *There is a correspondence between the rooted binary trees and the parenthesized expressions.*

Proof See [57], pages 51-52.

The authors show that these are equivalent using induction on the number of vertices. We can also see C_n is the cardinality of the set of rooted binary trees by appealing to a general rooted binary tree T with subtrees T' and T".

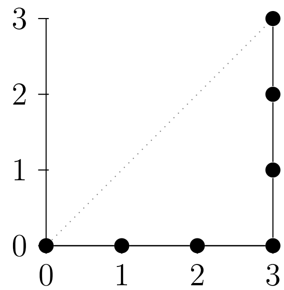


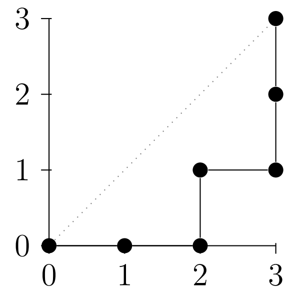
From this image, we can see that the combinatorial method of counting the rooted binary trees gives rise to the recurrence relation for the Catalan numbers inductively.

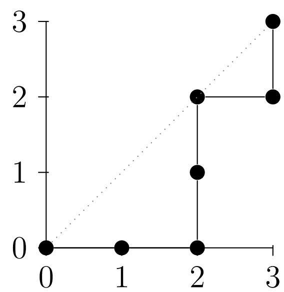
3.5 Monotonic Paths

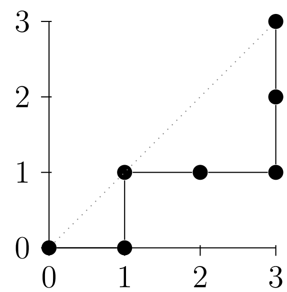
The monotonic paths share a close relation to the full rooted binary trees. We begin by generating all of the monotonic paths on a 3×3 grid, and then show the connection to the rooted binary trees.

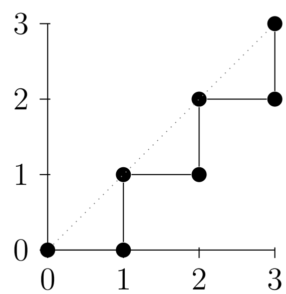
Here are all of the monotonic paths on a 3×3 grid:











The fact that there are five such monotonic paths leads to the following proposition:

Proposition 3.5.1 *There is a bijective correspondence between the rooted binary trees and the set of monotonic paths that do not cross the diagonal.*

Proof See [58], page 59-61. ∎

In Stanton and White's text [58], they generate a string of parenthesized expressions from a tree, and then from the parenthesized expressions generate a monotonic path. Begin by "pruning" the lowest children leaves off the tree and writing them down in order from left to right. Then, write down the parents of those children after. The following diagram illustrates how this is done:

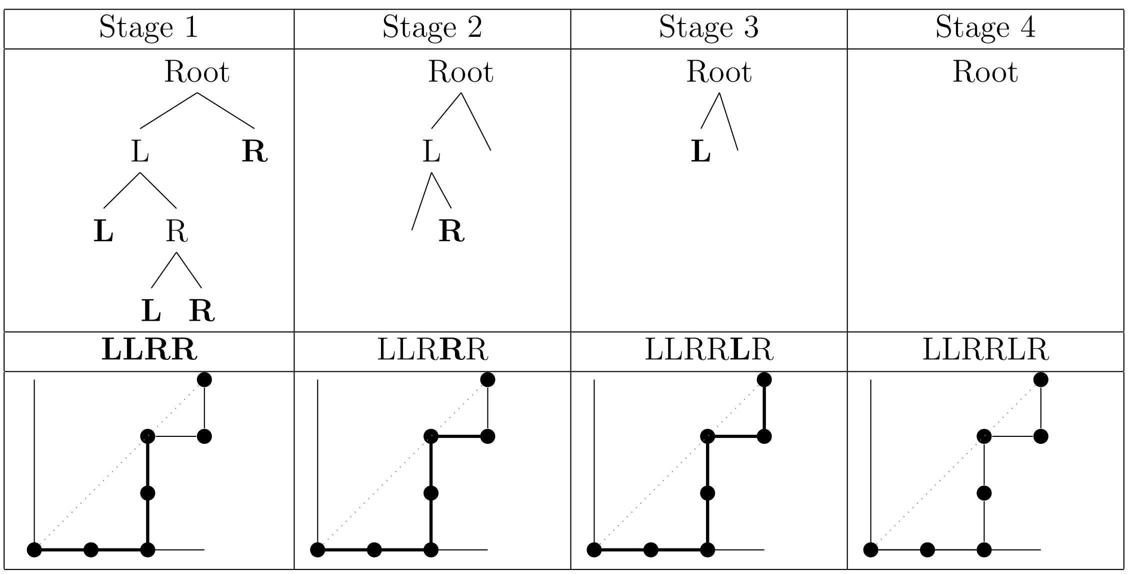


Figure 3.3: Binary Trees and Monotonic Paths Correspondence

3.6 The Reflection Principle

In this section we show the connection between the monotonic paths and the reflection principle. For this particular interpretation, we follow Feller's exposition.

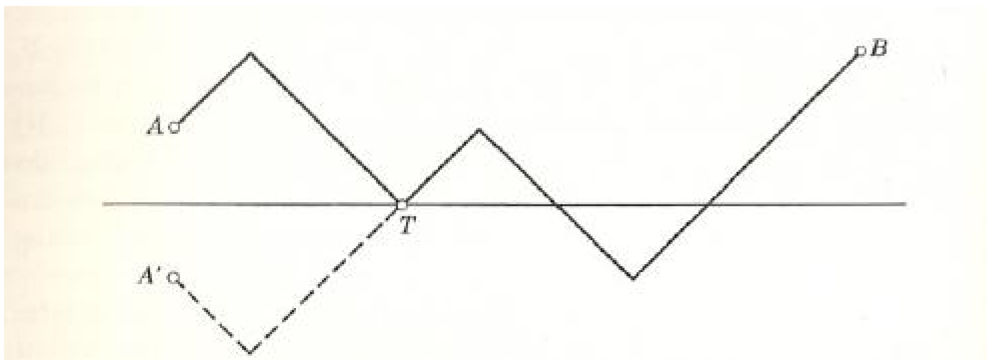


Figure 3.4: Reflection Principle [85], page 72.

Let $A = (a, \alpha)$ and $B = (b, \beta)$ be the points in the positive quadrant: $b > a \geq 0$, $\alpha > 0$, $\beta > 0$. By reflection of A on the t-axis is meant the point $A' = (a, -\alpha)$ (see 3.6). A path from 'A' to 'B' is defined in the obvious manner [85], page 72.

Proposition 3.6.1 ([85], page 72) *The number of paths from A to B which touch or cross the x-axis equals the number of paths from A' to B.*

Proof Feller's probability text has an excellent exposition on the reflection principle [85], page 72-73. ∎

Feller uses the above image 3.6 to show that every path that touches or crosses the axis has a matching reflection which also touches or crosses the axis. Using this proposition, we can remove the number of bad paths from the total number of paths, and we obtain a difference of coefficients: $\binom{2n}{n} - \binom{2n}{n+1}$. Feller does this in a more general setting for the ballot problem. This method is based off of the same principle first introduced by André, as mentioned before in 1.1, where he uses a similar idea to count the number of bad paths.

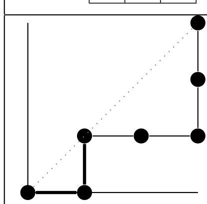
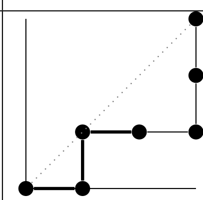
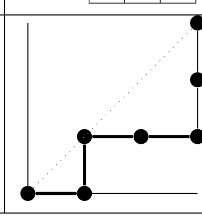
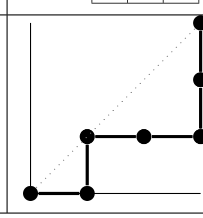
3.7 Young Diagrams

A Young diagram (or Ferrer's diagram) is a combinatorial structure used in the theory of integer partitions. Since there are two standard Young tableaux of shape $(2, 2)$, and five such tableaux of shape $(3, 3)$, we expect:

Proposition 3.7.1 *There is a bijective correspondence between the monotonic paths from* $(0, 0)$ *to* (n, n) *that do not cross the diagonal and standard Young tableaux of shape* (n, n)*.*

Proof Stanton and White's combinatorics text has a correspondence [58], page 62.
∎

The correspondence between the monotonic paths and the standard Young tableaux can be visualized quite nicely:

<table>
<tr><th>Step 1</th><th>Step 2</th><th>Step 3</th><th>Step 4</th></tr>
<tr>
<td>$\begin{array}{|c|c|c|}\hline 1 & & \\ \hline 2 & & \\ \hline\end{array}$</td>
<td>$\begin{array}{|c|c|c|}\hline 1 & 3 & \\ \hline 2 & & \\ \hline\end{array}$</td>
<td>$\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & & \\ \hline\end{array}$</td>
<td>$\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 & 6 \\ \hline\end{array}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</table>

At each step i, if the step is to the East, we place the number i in the first row. If the step is to the North, we place the number i in the second row of the tableaux. This process can also be reversed in a similar fashion. This same method was employed by Peter Striegel in his capstone project with Dr. Stockbridge and Dr. Willenbring in relating the Young tableaux to monotonic paths [23]. Their correlation also hints to a connection between the reflection principle for monotonic paths to a possible corresponding principle for the Young tableaux.

3.8 Schubert Calculus for the Grassmannian of 2-Dimensional Subspaces in $\mathbb{C}^n$

The Schubert calculus was introduced in the late nineteenth century as a means to answer some enumerative geometric problems in an algebraic setting. In this section we consider the problem: How many lines in $\mathbb{CP}^3$ intersect four generic lines? Recent literature provides a correspondence between these lines, and standard young tableaux of shape $(2, 2)$. The problem can be generalized.

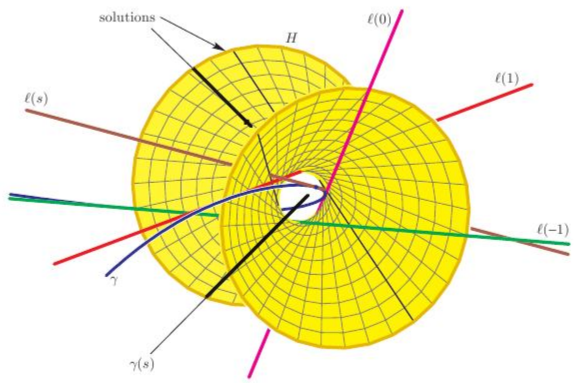
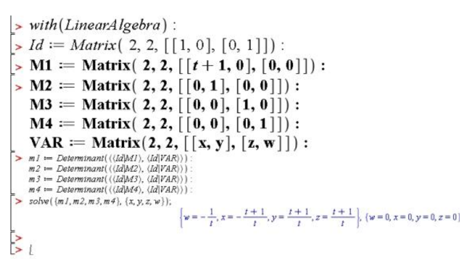


Figure 3.5: Example of Four Lines [41]

The example of the four lines in projective 3-space can be shown using the aid of a computer to find the number of solutions. A line in $\mathbb{CP}^3$ may be represented by a GL_2-equivalence class of a 2×4 full rank matrix. For the problem of four lines, the following maple code will find the number of lines coincident to four generic lines:



```
> with(LinearAlgebra) :
> Id := Matrix( 2, 2, [[1, 0], [0, 1]]) :
> M1 := Matrix( 2, 2, [[t + 1, 0], [0, 0]]) :
> M2 := Matrix( 2, 2, [[0, 1], [0, 0]]) :
  M3 := Matrix( 2, 2, [[0, 0], [1, 0]]) :
  M4 := Matrix( 2, 2, [[0, 0], [0, 1]]) :
  VAR := Matrix(2, 2, [[x, y], [z, w]]) :
> m1 := Determinant((<Id|M1>, <Id|VAR>)) :
  m2 := Determinant((<Id|M2>, <Id|VAR>)) :
  m3 := Determinant((<Id|M3>, <Id|VAR>)) :
  m4 := Determinant((<Id|M4>, <Id|VAR>)) :
> solve({m1,m2,m3,m4}, {x,y,z,w});
```

$$\left\{w = -\frac{1}{t}, x = -\frac{t+1}{t}, y = \frac{t+1}{t}, z = \frac{t+1}{t}\right\}, \{w = 0, x = 0, y = 0, z = 0\}$$

```
>
>
```

Figure 3.6: The 2 lines in $\mathbb{CP}^3$ meeting 4 generic lines

Generalizing the example of the two lines in $\mathbb{CP}^3$ that meet four generic lines, we seek to find the number of lines in $\mathbb{CP}^{n+1}$ that meet $2n$ fixed codimension 2 projective subspaces. One can see that in taking the determinant of 3.8, the variables in the 2×2 minor describe a quadratic surface. This surface can be seen in the image generated by Sottile (see 3.5).

Proposition 3.8.1 *There is a correspondence between the lines in $\mathbb{CP}^{n+1}$ that meet $2n$ fixed codimension 2 projective subspaces and the standard Young tableaux.*

Proof Schubert proves this explicitly [68]. This is shown in Fulton's text as well [82]. ∎

The Young diagram $\lambda = (\lambda_1, \lambda_2)$ with $\lambda_1 \leq n$ parameterize the cells in $\mathbb{G}_{2,n+2}$. If we let $\mathbb{X} = M_{2,n+2}(\mathbb{C})$, the map $GL_2 \times \mathbb{X} \to \mathbb{X}$ is a group action mapping $(g, m) \to gm$. Given $m \in \mathbb{X}$, the orbit of $m \in \mathbb{X}$ is specified by the reduced row echelon matrix of m. A reduced row echelon matrix is of the form:

$$R(\vec{a}, \vec{b}) = \begin{bmatrix} 0 & \dots & 0 & 1 & a_1 & \dots & a_k & 0 & a_{k+1} & \dots & a_{k+l} \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 & 1 & b_1 & \dots & b_l \end{bmatrix}$$

Thus we have a disjoint union:

$$GL_2 \setminus \mathbb{X} = \bigsqcup_{\substack{k,l \geq 0 \\ l+k \leq n}} \left\{ \left[R(\vec{a}, \vec{b}) \right] | \vec{a} \in \mathbb{C}^{k+l}, \vec{b} \in \mathbb{C}^l \right\}$$

Thus the orbits in $\mathbb{X}$ may be partitioned into subsets indexed by $\lambda = (\lambda_1, \lambda_2)$ such that $\lambda_1 = k + l$ and $\lambda_2 = l$

3.9 Branched Coverings

In this section we discuss the interpretation of the Catalan numbers as they pertain to the branch sets of complex rational functions. We exhibit a function between the branch set of a rational map and the lines in $\mathbb{CP}^{n+1}$ that meet $2n$ generic fixed codimension 2 projective subspaces.

Proposition 3.9.1 *There is a correspondence between the lines in $\mathbb{CP}^{n+1}$ that meet $2n$ fixed codimension 2 projective subspaces and the branch set of a rational map whose critical points are fixed.*

Proof Goldberg provides exposition [39]. See Erekmenko and Gabrielov for the explicit map between the two [38]. $\blacksquare$

To close the final loop, we must show that the translation conjugacy classes of degree d monic polynomials in one complex variable, whose critical points are fixed, is exactly C_{n+1}.

See [7] for the **degree** of a variety.

Proposition 3.9.2 (See [39]) *The degree of the Plucker embedding $E_{n+1} : G_2(Poly_{(n+1)}) \to \mathbb{P}^{\frac{(n+1)(n+2)}{2}-1}$ is $\rho(n+1)$, where $\rho(n+1) = \frac{1}{n+1}\binom{2n}{n}$ is the Catalan number.*

Proof Goldberg provides a proof in [39]. Eremenko and Gabrielov also have a great exposition furthering her results [38], and Sottile has great imagery with his discourse [41]. $\blacksquare$

There is a many similarities between the branched coverings and the Schubert calculus interpretations. On one hand, we have a complex rational function with a $GL(2,\mathbb{C})$, fractional linear transformation, acting on the left. On the other hand we have a $2 \times n$ matrix with a $GL(2,\mathbb{C})$ acting on the left. Since these two are closely related, the mapping between these two ideas can be given explicitly:

$$\frac{a_0+a_1x+\cdots+a_nx^n}{b_0+b_1x+\cdots+b_nx^n} \leftrightarrows \begin{bmatrix} a_0 & a_1 & \dots & a_n \\ b_0 & b_1 & \dots & b_n \end{bmatrix}$$

If we want to reformulate the complex rational function into the Grassmannian interpretation, we strip the coefficients and encode them in a $2 \times d$ matrix similar to the one above. Conversely, we need only take the matrix entries and make them coefficients of a complex rational function in the natural way.

3.10 Diagonal Harmonics of the Symmetric Group

Given a group G acting linearly on a vector space V. The ring, R, of complex-valued, polynomial functions on V has a subspace of *G-Harmonic* functions, which are defined as the solutions to constant coefficient G-invariant differential operators on R. In the special case when V is n dimensional euclidean space and $G = O(n)$ (the orthogonal group), the G-harmonic functions correspond to the spherical harmonics (i.e.: solutions to the Laplace equation). One can replace $O(n)$ with any group to obtain a generalization of spherical harmonics. We consider the case where G is the symmetric group, S_n.

If we let R denote the commutative algebra of polynomials in n variables over the complex numbers, and I the ideal generated by the non-constant, symmetric polynomials then the quotient ring, R/I, is isomorphic to the complex cohomology ring of the flag variety. The S_n-harmonic polynomials are equivalent, as a graded S_n-representation, to R/I. The flag variety has the property that it is a homogeneous projective space that surjects onto each Grassmannian of $\mathbb{C}^n$. Thus, there is a abstract connection between the Schubert calculus and S_n-harmonic polynomials.

If we instead consider polynomials in $2n$ variables and mod out by the "diagonal" S_n-invariants one obtains the ring of "diagonal harmonic" polynomials introduced by Adriano Garsia and Mark Haiman. It would be nice, following the scheme of the thesis thus far, if there was a correspondence between the branched coverings and diagonal harmonics. The authors actually showed the isotypic component of the sign representation has dimension equal to the Catalan number [24]. These conjectures were proved in 2002 (see [5]).

Proposition 3.10.1 (See [24], page 31) *The Frobenius series of T_n is $q^{\frac{n}{2}}\omega\mathfrak{F}_n(q, q^{-1})$, where ω is the canonical involution on symmetric functions corresponding to tensoring with the sign character. The Hilbert series of U_n is $q^{\frac{n}{2}}C_n(q, q^{-1})$, where C_n is the conjectured t,q-Catalan number defined by $C_n(t, q) = \langle s_{(1^n)}\mathfrak{F}_n(t, q)\rangle$.*

Proof See Haiman's 2002 proof in [5]. $\blacksquare$

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