Wave Engineering Lab

Particle Trajectories in Deep & Intermediate Depths

Mass Transport Under Waves

Master SMA

Basics of hydrodynamics

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Objective: set-up a program simulating particle trajectories under regular 2D waves in linear theory.

First we are going to give the solution to the linearized problem for waves, which is solved using the method of separation of variables. This is the solution to a 2D problem for modelling waves and so we will work on the plane XZ.

First we have to take into account that the application of this solution is restricted to low steepness waves. Steepness is defined as the relation between the amplitude of the wave and the wavelength (Eq.1). Steepness is a parameter driving the nonlinear effects and so we want it to be as low as possible since the linearized problem does not take into account nonlinear effects. Therefore we are going to use in this practical exercise waves with a low steepness.

Eq. 1

The wave form is given by Eq. 2.

( ) Eq. 2

The velocity potential is given by Eq. 3.

( ) ( ) Eq. 3

The velocity field can be found by differentiation of velocity potential (Eq. 4 and Eq. 5).

( ) ( ) Eq. 4

( ) ( ) Eq. 5

Moreover the solution gives us a relation between the period and the wavelength depending on water depth. This relation is called the dispersion relation (Eq. 6).

( ) Eq. 6

To describe the trajectories of the particles we are going to use the solution of the linearized problem above. We have calculated and represented the trajectories for different depths to see the influence of the sea floor. We have established a fixed wavelength (λ) and fixed amplitude (a) and we have represented the influence of the sea floor by changing the depth (h) for each case. The period (T) has been obtained by the dispersion relation that relates the period to the wavelength and the water depth.

We have used a wavelength of λ=150 m which can represent a wave caused by wind (60 - 150 m) and an amplitude of 1m. The amplitude of 1m has been chosen in order to have a low steepness. In fact the steepness with this values for the amplitude and the wavelength is . The three cases presented on the lecture notes, which we want to represent, are defined on Fig. 1a and Fig. 1b.

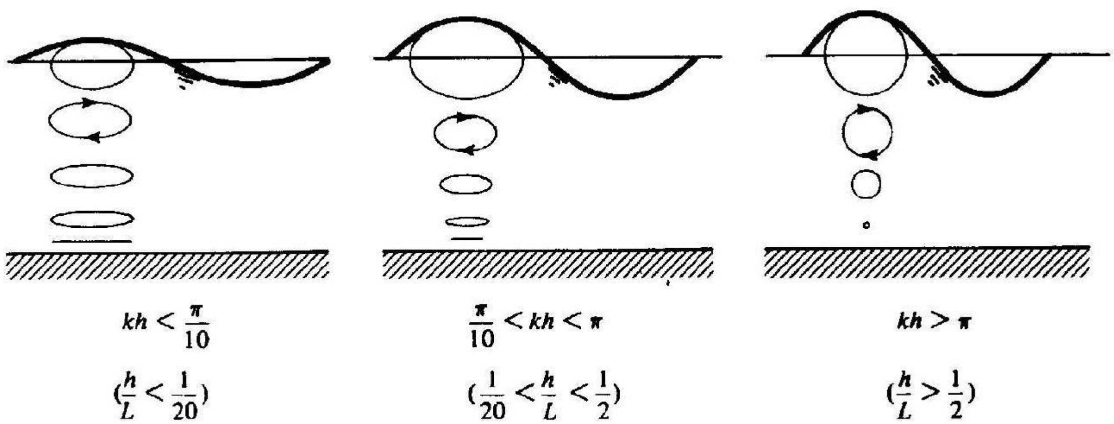

Fig. 1a Influence of the water depth in comparison with the wavelength

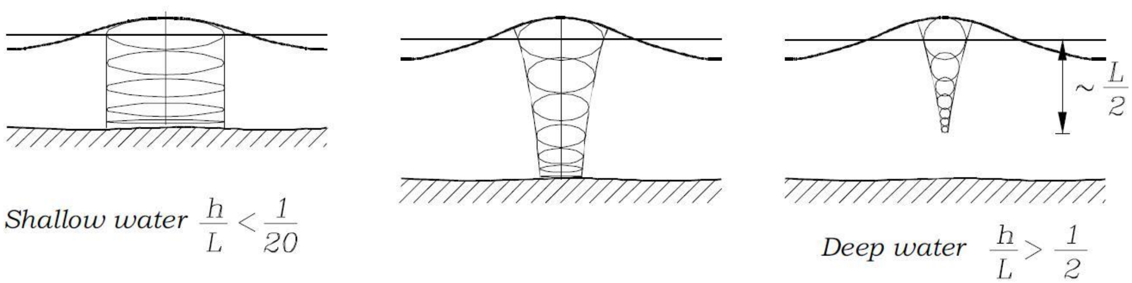

Fig.1b Influence of the water depth in comparison with the wavelength

To represent shallow water we have used a depth of 5 m. If h=5m, – — ˆ — .
Therefore we are in the case of shallow water.

To represent deep water we have used a depth of 100 m. If h=100 m, – — ˆ – .
Therefore we are in the case of deep water.
To represent the intermediate case, we have used a depth of 25 m. If h=25m, – — ˆand
. Therefore we are in the intermediate case.

A- Velocities estimated at the mean particle position. Your simulation should recover results given in Lecture Notes, i.e. closed trajectories.

The matlab code related to this part is trajectories1Np.m

On this part of the practice we are going to represent the trajectories using the velocities estimated at the mean particle position. This means that the velocity changes on time but not due to the new position of the particle. To calculate the trajectories we integrate the velocity (Eq. 4 and Eq. 5) which leads us to Eq. 7 and Eq. 8.

( ) ∫ ( ) + C1 Eq. 7

( ) + C2 Eq. 8

As we have said before the velocities will only depend on the mean particle position, therefore if we call x_m and z_m the mean particle position, the position on each time will be defined as in Eq. 9 and Eq. 10. Notice that we have to constants C1 and C2, due to the integration. We have fixed them to C1=x_m and C2=z_m. If we do so, we have the mean position on the center of the ellipse described by the trajectory.

( ) ∫ ( ) + Eq. 9

( ) ∫ ( ) + Eq. 10

Now the new position (Eq. 9 and Eq. 10) on each time only depends on the mean particle position and the time.

To represent the trajectories we have chosen a number of points equally distributed on depth (z_m), and we have discretized the trajectory of each point on time. We have taken N points of discretization on time, –. Eq. 9 and Eq. 10 have been evaluated at each time step . In all three cases we have used 2000 points of discretization to have a good resolution of the trajectory and without taking a lot of time for calculation. X_m and have been fixed to 0 for all the three cases for simplicity.

First case: Deep water

The first case which represents deep water uses the parameters given on table 1.

λ	T	h	a	ε	h/L
150 m	9,8 s	100 m	1 m		

Table 1. Parameters for deep water

The trajectories are represented up to less than half of the depth due to the fact that the trajectories below are difficult to see because they are too small in comparison to the scale of z_m which goes from 0 to -100 (Fig. 2). We can see that the trajectories are circular and their diameter decreases with the depth. At the surface the radius is naturally equal to the amplitude of the wave, as it can be seen on fig.2.

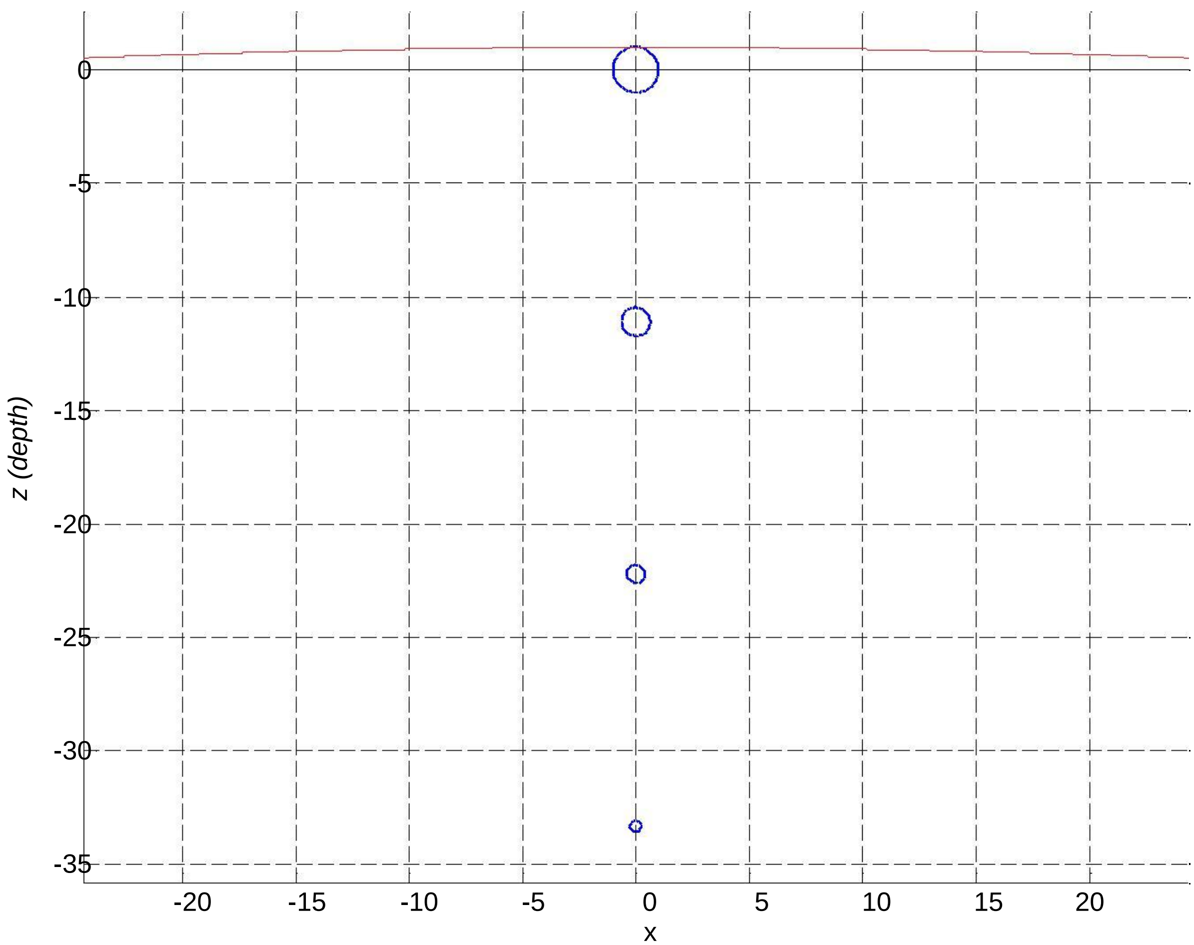

Fig. 2 Trajectories in deep water

Second case: Between deep water and shallow water

The second case which represents the intermediate case between shallow water and deep water uses the parameters given on table 2.

λ	T	h	a	ε	h/L
150 m	11,1 s	25 m	1 m		

Table 2. Parameters for deep water

The trajectories are represented on Fig. 5. In this case we can see that the trajectories are not circular anymore, now they are elliptic. We can see that the deeper the point we consider the more flat the elliptic trajectory is, until we arrive to the bottom (z_m=-25m) where the trajectory is a line.

We can compare the solution of the trajectories presented on Fig. 5 with the formulas given on the lecture notes (Fig. 3). In table 3 the results from the formulas given on the lecture notes for different points are presented and can be compared with Fig. 4 which is the same as Fig 5 but with zoom (Fig. 4a for z_m=0 m, Fig. 4b for z_m=-12,5 m and Fig.4c for z_m=-25 m).

Z_m	A [m]	B [m]
0	1.28	1
11.11	0.94	0.49
25	0.8	0

Table 3

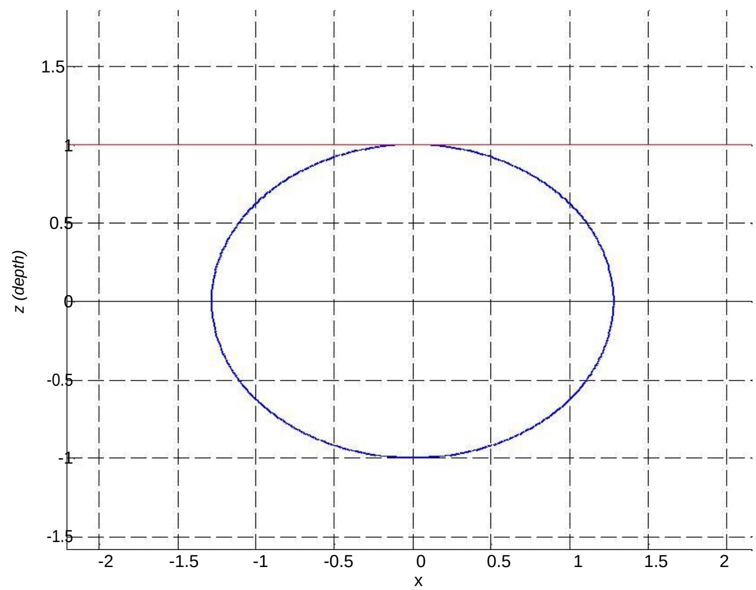

Fig. 4a Trajectory for z_m=0m

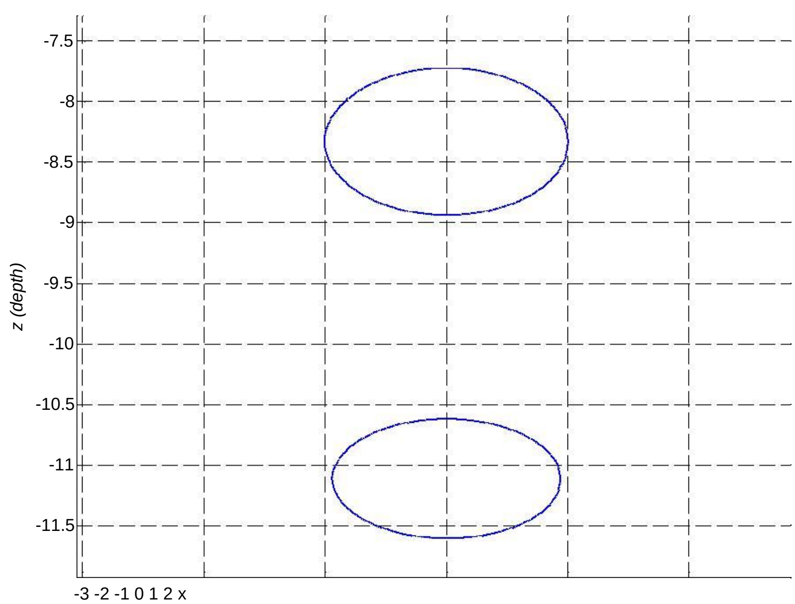

Fig. 4b Trajectory for z_m=-11.11m

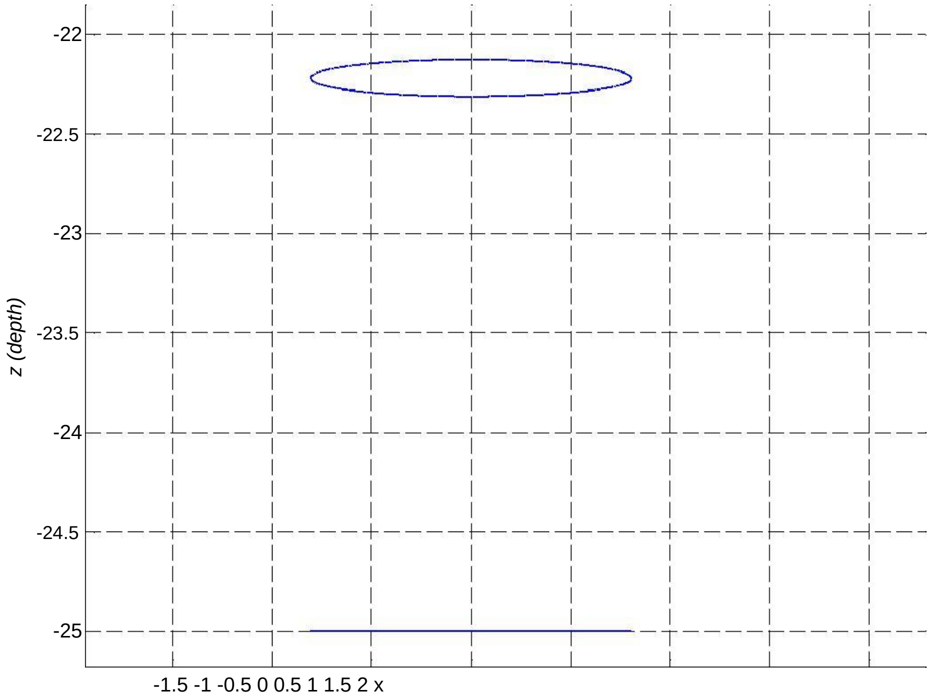

Fig. 4c Trajectory for z_m=-25m

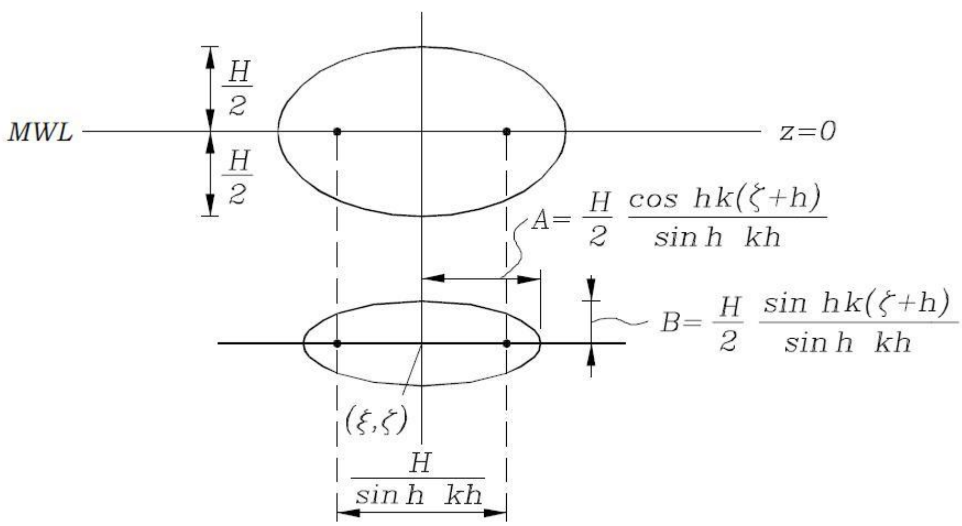

Fig. 3 Particles paths

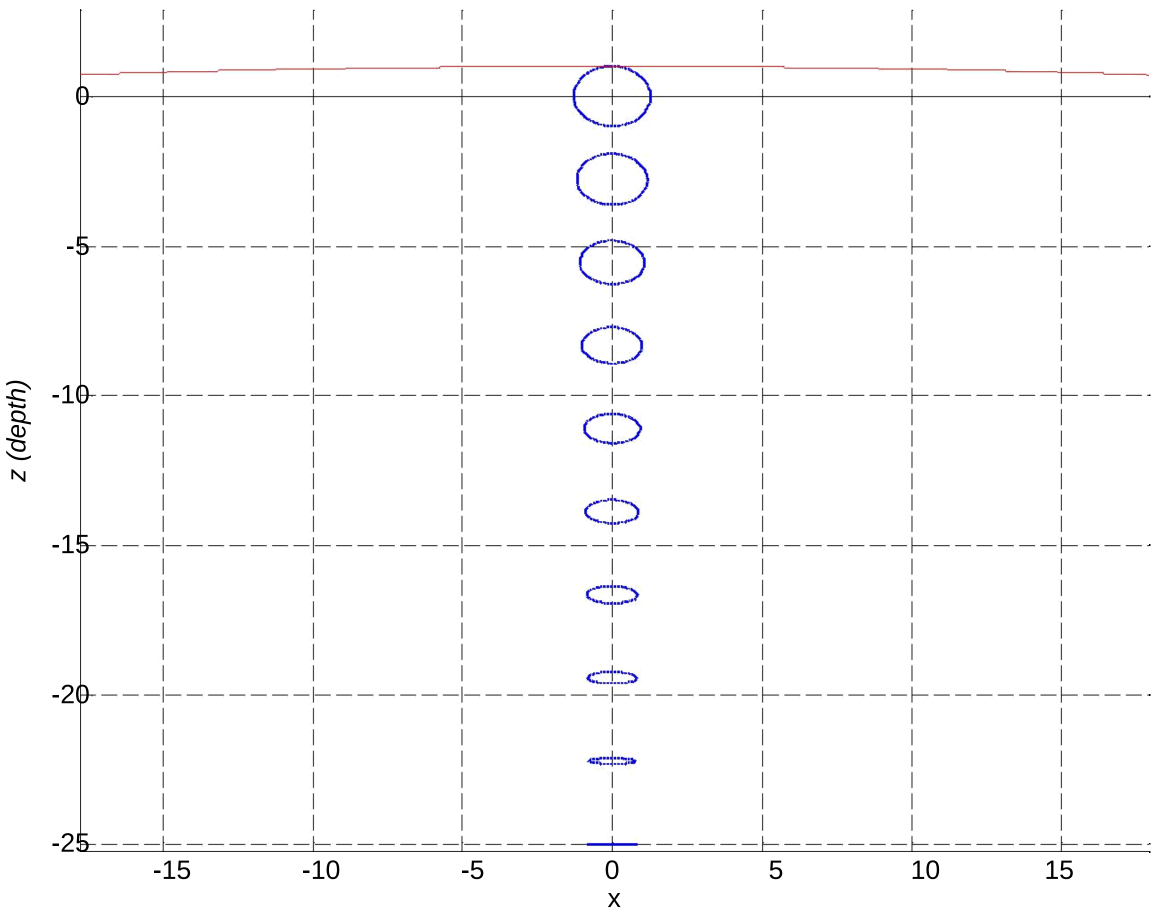

Fig. 4 Trajectories for the second case

Third case: shallow water

The third case which represents shallow water uses the parameters given on table 4.

λ	T	h	a	ε	h/L
150 m	21.6 s	5 m	1 m		

Table 4. Parameters for deep water

The trajectories are presented on Fig. 6. In this case we have elliptical trajectories that they become flatter with the depth until for z_m=-h (z_m=-5m) where the trajectory is a line. However their widths do not almost depend on the depth, they are constant with the depth, whereas on the two first cases the width diminished with depth.

This width, which is almost 9.5 m in this case, can be explained by the formulas given before (Fig. 3) and it is given below:

( )

The simplification of ( ) to 1 is done because and if we take z_m different form 0, ( ) will be closer to 1. It is for that reason that the width of the trajectory barely depends on the depth of the mean position (z_m).

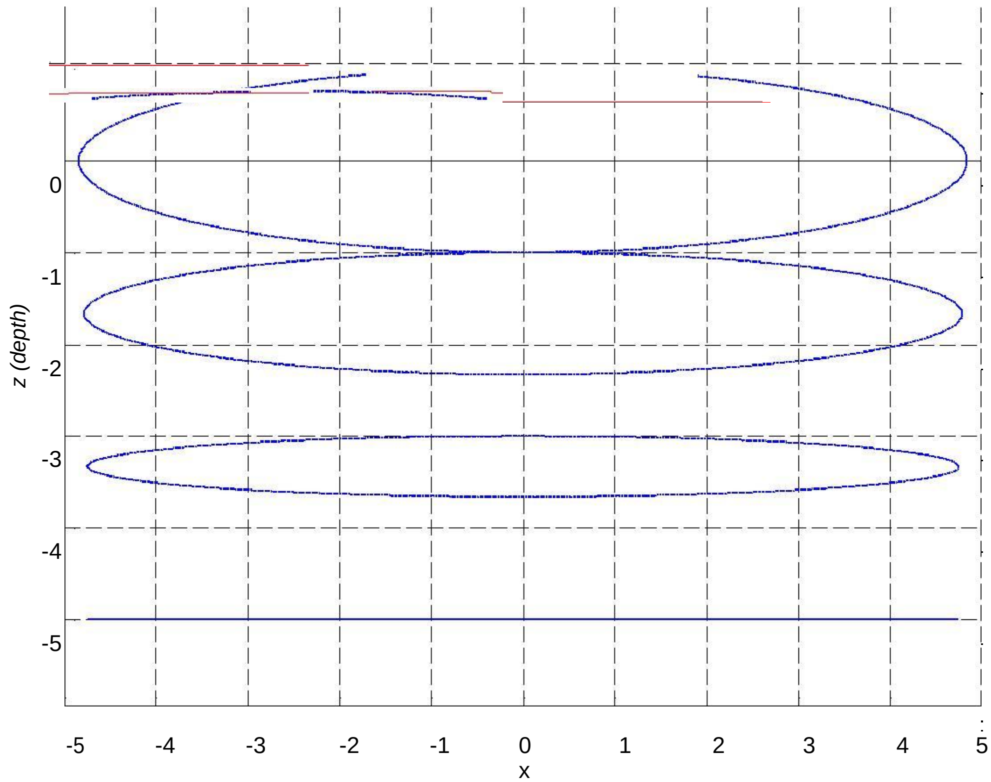

Fig. 6 Trajectories for shallow water

From this part, we have seen that increasing the water depth makes the solution less dependent from the water depth. With low water depth (shallow water) the trajectories are elliptical and with the same width. For high water depth (deep water) the trajectories are almost circular and with decreasing radius with depth. For intermediate depths we have an intermediate of both solutions, elliptical trajectories whose width decrease with depth (as in deep water) and that are flatter the deeper they are.

We can observe as well that for a given wavelength, the lower is the water depth the wider are the trajectories.

B- Velocities estimated at the actual position

The matlab code related to this part is trajectories2Np.m

In this part the trajectories will be calculated using the velocity estimated at the particle position. This means that the velocity will depend on the time and on the actual position of the particle.

To calculate the trajectory of a particle located on a point with coordinates x_o, y_o we are going to discretised as well on the time. We take N points of discretization for a period T, and so . In each time step the position will be the position on the step before plus the velocity multiplied by the difference of time dt (Eq. 11 and Eq. 12).

( ) ( ) Eq. 11

( ) ( ) Eq. 12

Notice that in this case the velocity and the new position depends on the actual position and the time, and so ( ) ( ).

For this part we are going to consider a time of ten periods, since in this case the trajectory is not closed. We have found ten periods a good choice to see the trajectory.

The parameters used on each case are the same as in the previous part of the practice. Xo has been fixed to 0 for simplicity and in order to have the point on the wave surface at z=0. We want that because in that case z_o is almost the mean particle position of the trajectory, and it will be used to calculate the theoretical mean drift velocity.

First case: Deep water

The parameters used are presented on table 1.

The trajectories are represented in Fig. 7. Notice that the graph does not include all the depth (-100m) in order to be able to see the trajectories. We can see now that the trajectories are not closed trajectories and they present a drifting motion in the direction of the wave. The trajectories seem to be circular with a drifting motion. We can see as well that the drifting velocity decreases with the depth.

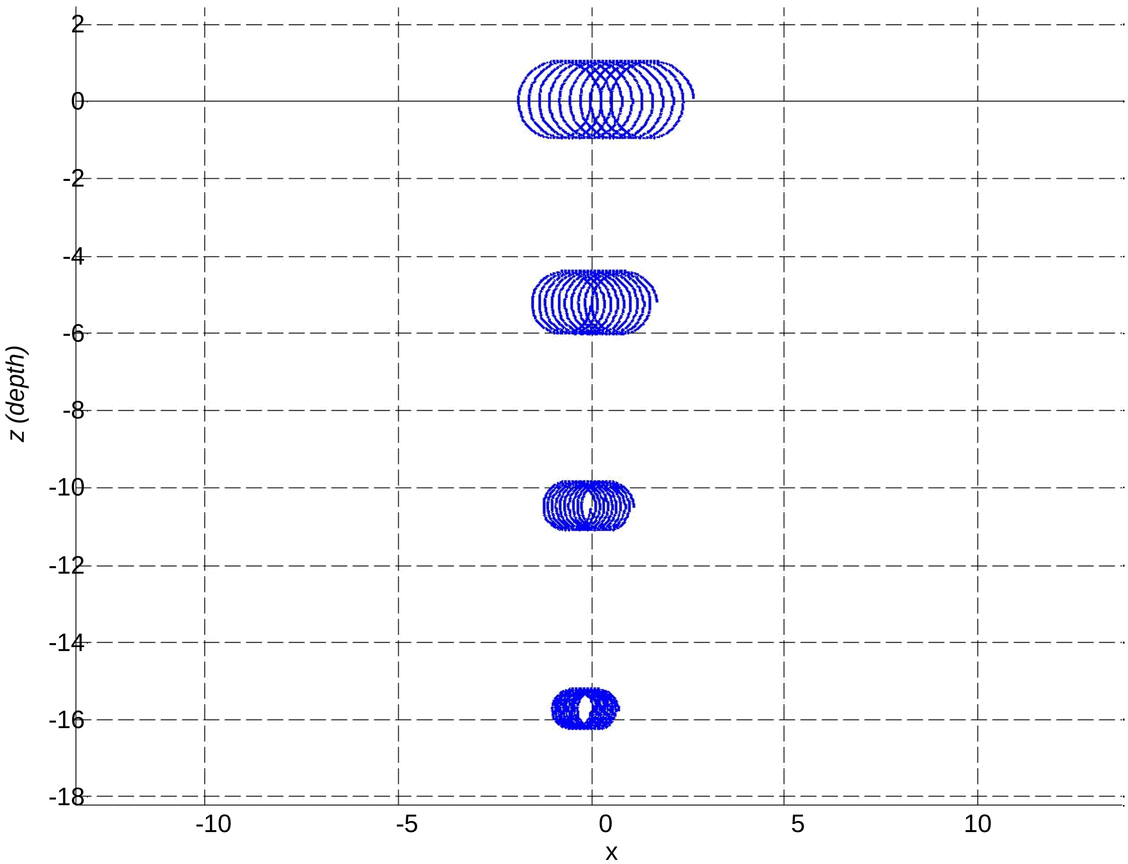

Fig. 7 Trajectories for deep water

Second case: Between deep water and shallow water

The second case which represents the intermediate case between shallow water and deep water uses the parameters given on table 2.

The trajectories are represented on Fig. 8a and Fig. 8b. We can see the same phenomena that in the first case but with a drifting motion: the trajectories are not circular anymore, now they are elliptic (Fig. 8b) with a drifting motion. We can see that the deeper the point we consider the more flat the elliptic trajectory is, until we arrive to the bottom (z_m=-25m) where the trajectory is a line.

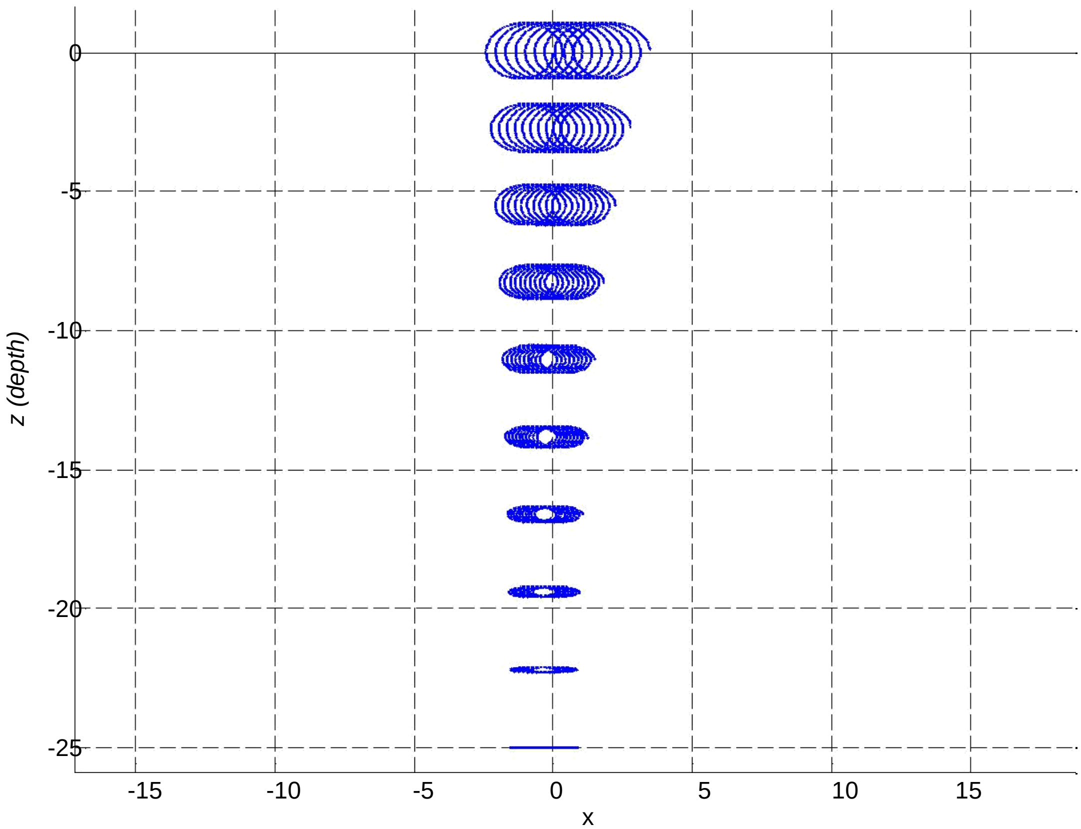

Fig. 8a Trajectories for the intermediate case

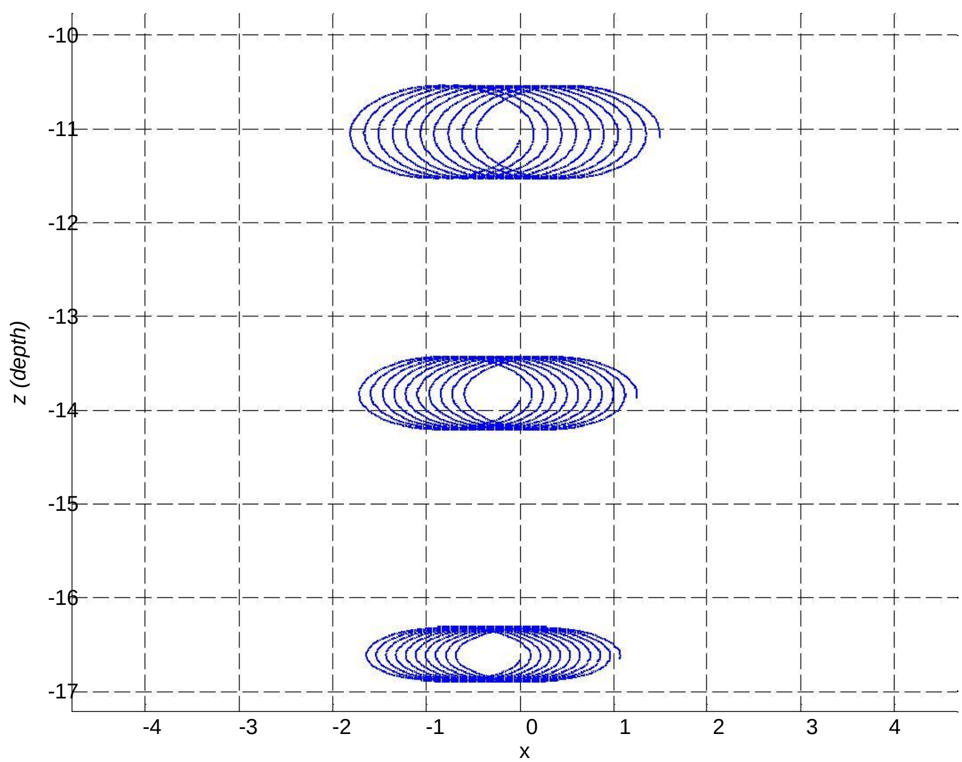

Fig. 8a Trajectories for the intermediate case

Third case: shallow water

The third case which represents shallow water uses the parameters given on table 4.

The trajectories are presented on Fig. 9. In this case we have elliptical trajectories with a drifting motion. The trajectories as on the first part of the practise become flatter with the depth until for z_o=-h (z_o=-5m) when the trajectory is a line. We can see that in this case the drifting velocity does not depend too much with the depth than in the previous cases as the three trajectories presented on Fig. 9 end almost at the same value of x. That is because the drift velocity depends on z by (( )) and again for kh very low this is almost equal to one.

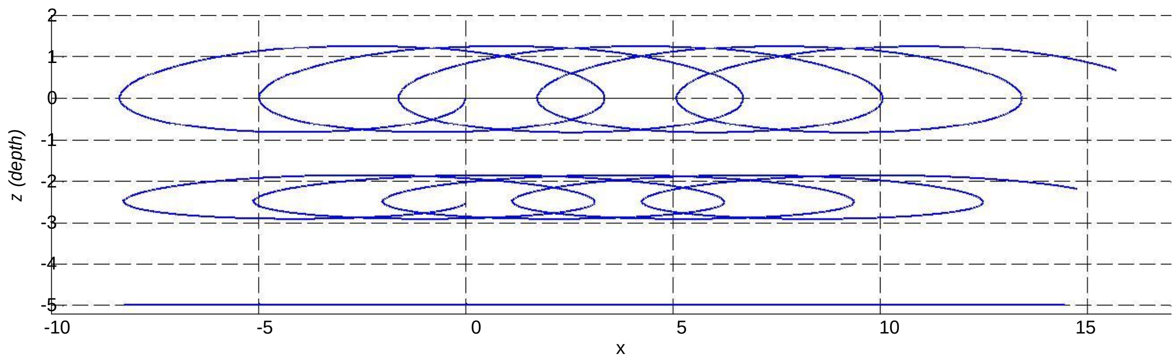

Fig. 9 Trajectories for shallow water

For this second part we have seen the same behaviour of the first part for each of the three cases but adding a drift motion. We have seen that a drift motion appears and that the drift velocity increases with the depth (maximum on the surface and minimum on the sea bottom) except on the shallow water where it does almost not depend on the depth.

We can see as well on the three cases that the period changes for each trajectory. We mean that after 10 periods the particle does not end at the same z, and so the period is not T. This is due to the fact that the period T is defined in an Eulerian form, we take a fixed point on the domain and we will see that the period is T. However if we followed a particle, the period changes because we are working with the Lagrangian form. In fact the period increases, we observe that the trajectories do not reach the end point that would reach if they had a period T.

ANIMATED GRAPHS

To better understand the motion of a particle under waves we have done three animations with matlab that can be found in the zip together with this report and the codes. For that purpose we have created the code trajectories_movie.m. We will refer as well to Fig. 10a and Fig. 10b where a plot of the velocity field is done.

For the three cases an amplitude of 0,5 m for the wave and a water depth of 10 m has been chosen in order to be able to see the wave shape in the case of deep water and understand why the trajectory is like defined in the previous part. We will change the wavelength for each case.

Dioh_Feliu_Ravi_1.avi represents the trajectories under waves of 20 m of wavelength. We can see with this value we represent the case of deep water. For the second case (Dioh_Feliu_Ravi_2.avi) we have taken a wavelength of 60 m, and we are between shallow water and deep water. Finally for the third case (Dioh_Feliu_Ravi_3.avi) the wavelength has been set to 200 m, which represents shallow water.

With these animated plots we can better understand the trajectories of the particles. These circular and elliptical movements is due to velocity field under a wave which is represented on Fig.10a and Fig.10b for a given time plus the fact that the wave moves and therefore the velocity field under the wave changes. We can see that the velocity decreases with the depth and this is why the trajectories for deeper particles are smaller. With the animated plots we can see how the velocity field changes with time and with the movement of the wav. With that non-stationary velocity field we see how the trajectory develops. We can see that the velocity vector in each moment is tangential to the new position of that moment.

We see as well that for the trajectory of a particle on the surface, this particle remains on the surface. This is very important, because the particles on the surface must remain always on the surface as stated by the problem. For the case of shallow water (Dioh_Feliu_Ravi_1.avi) we see that the particle of the surface is always almost on the wave surface, there is a bit of error, which is due to the fact that the steepness is not very low. If we decrease the steepness (as done in the other two animations) the difference between the particle position and the wave surface decreases, becoming almost zero, non-perceptible. We have linearized the kinematic boundary condition to obtain the solution, and this is what is causing this error, which decreases with the steepness.

We can see as well why the trajectories are different in each case. For deep water we see that the velocity decreases with depth very quickly, that is what makes the trajectories smaller when deeper. For shallow water we see that the horizontal component of the velocity is very constant with depth, what makes that the trajectories can be flatter with depth but not less wide.

Finally we would like to comment that when the particle is under the trough of the wave the particle is pull back and when it is under a crest it is push forward. This can be compared with experiences on the beach. When you are on the beach and a wave crest is behind you, you are pulled back to the crest, and when it reaches you, the wave push you forward.

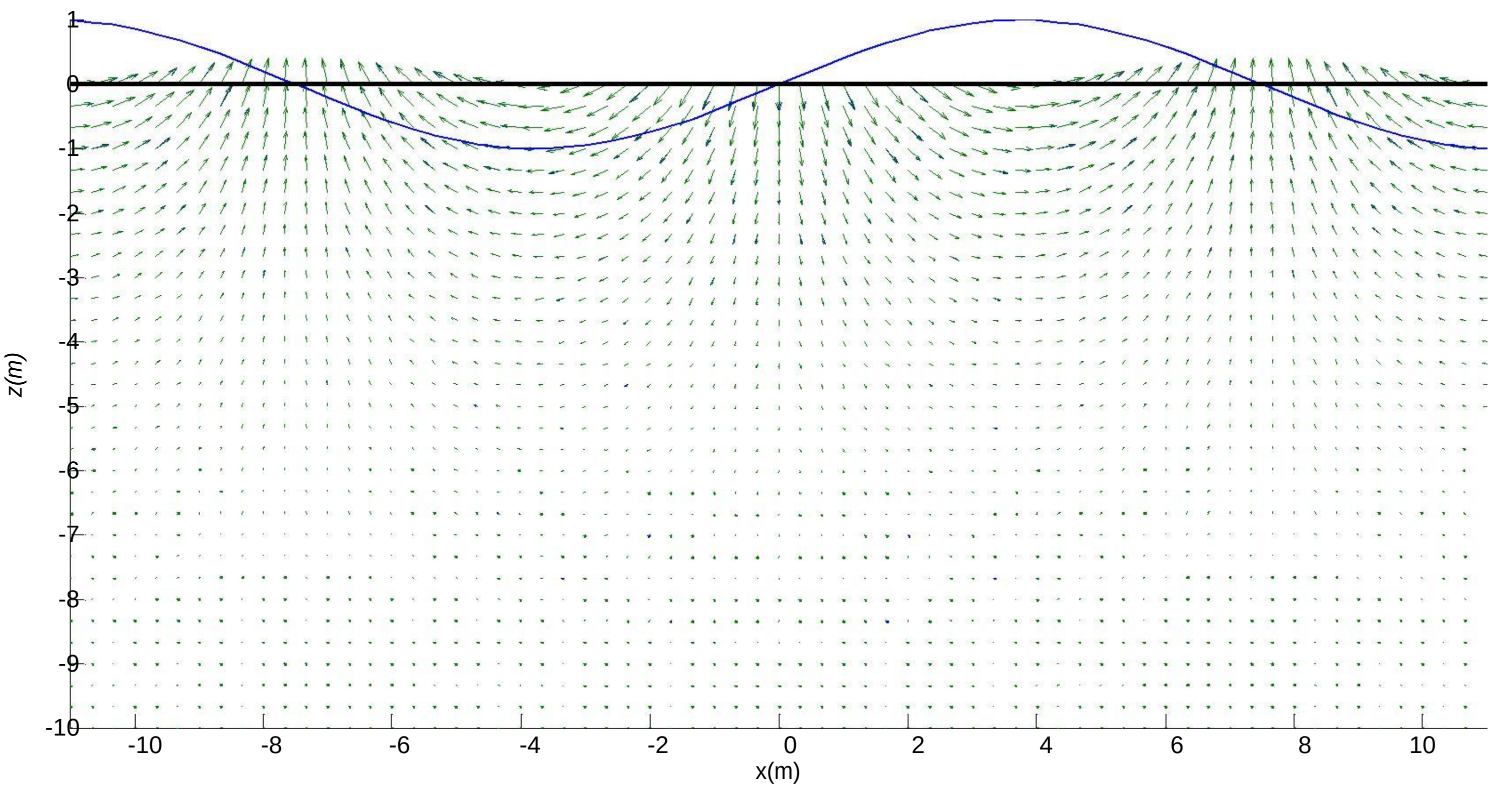

Fig. 10a Velocity field (deep water)

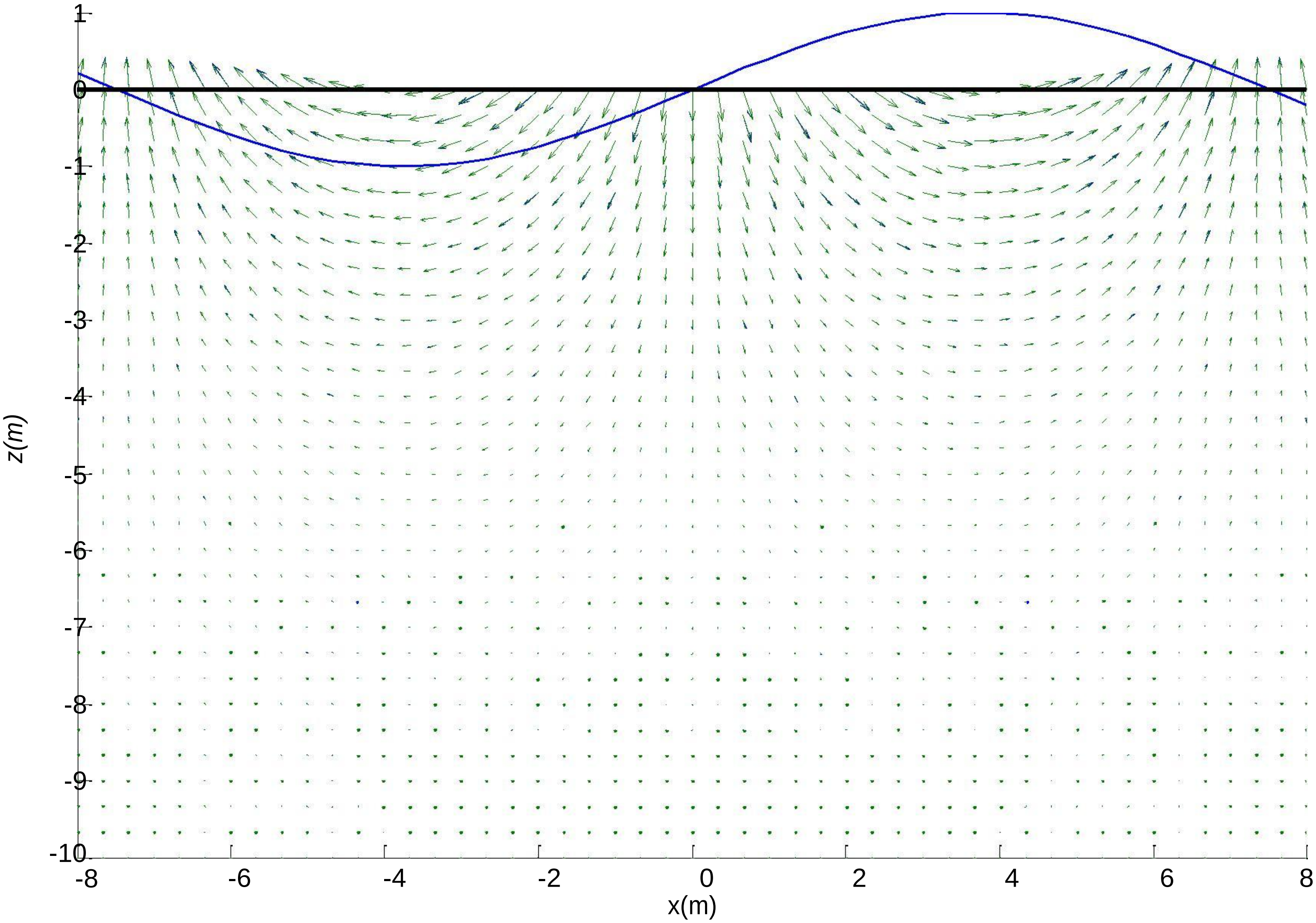

Fig. 10b Velocity field (deep water)

DRIFTING VELOCITIES AND MASS TRANSPORT

For this part the code used is trajectories2Np.m and meanvelocity.m

We have seen that a drifting motion appears; now we are going to calculate the mean drift velocity at the free surface and the domain for different wave amplitudes.

The mean drift velocity is estimated by taking the horizontal distance between the point on the beginning and the point at the end of the path of the particle divided by time. We compared the numerical values and theoretical one given by the mean velocity formula (Eq. 13).

( ) (( )) ( ) Eq. 13

The z used in Eq. 13 is the mean value of z for each trajectory, what means the centre of the trajectory, and as we have said before is equal to z_o as we have defined such that by taking –

Integrating mean drift velocity along the depth (z) and multiplying it by the density we obtain the mass transport, which can be compared to the theoretical mass transport given by Eq. 14. The mass transport is expressed in kg/m per unit of width.

——— Eq. 14

Next, the theoretical and numerical mean drift velocity along the depth is presented for each case. As well as the mass transport.

First case: Deep water

The first case which represents deep water uses the parameters given on table 1.

On Fig. 11 we can see how the mean drift velocity decreases very quickly with the depth and that the numerical and the theoretical results are quite close.

On table 5 we present the mass transport calculated in three ways. Using Eq. 14 (Mt), using the theoretical velocities (Mt_2) and numerically (Mnum). Mnum has been calculated for different number of points (N) along z.

	N=100	N=1000
Mt [kg/s]	0. 3292	0.3292
Mt_2 [kg/s]	0.3434	0.3306
Mnum [kg/s]	0.3485	0.3415

Table 5

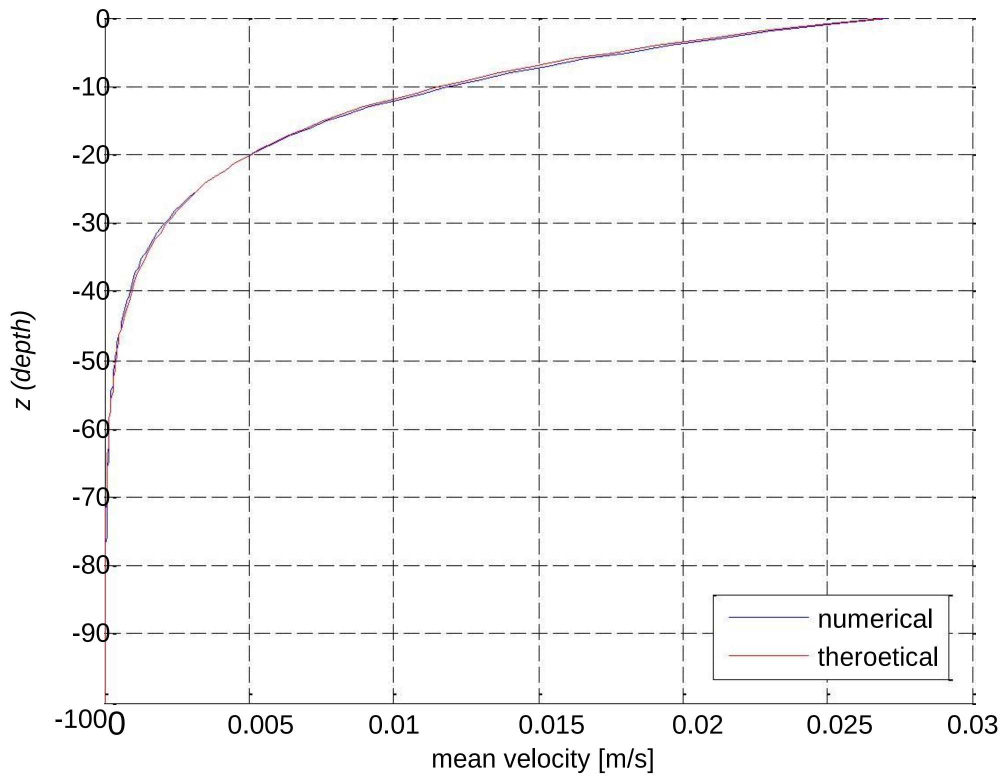

Fig. 11 Mean drift velocity for different depths in the case of deep water

Second case: Between deep water and shallow water

The second case which represents the intermediate case between shallow water and deep water uses the parameters given on table 2.

On Fig. 12 we can see how the mean drift velocity changes with the depth and that the numerical and the theoretical results are quite close. In this case, the mean drift velocity decreases as well with the depth but not as quickly as in the first case.

On table 6 we present the mass transport calculated in three ways. Using Eq. 14 (Mt), using the theoretical velocities (Mt_2) and numerically (Mnum). Mnum has been calculated for different number of points (N) along z.

	N=25	N=100
Mt [kg/s]	0. 3725	0. 3725
Mt_2 [kg/s]	0.3931	0.3730
Mnum [kg/s]	0.3936	0.3770

Table 6

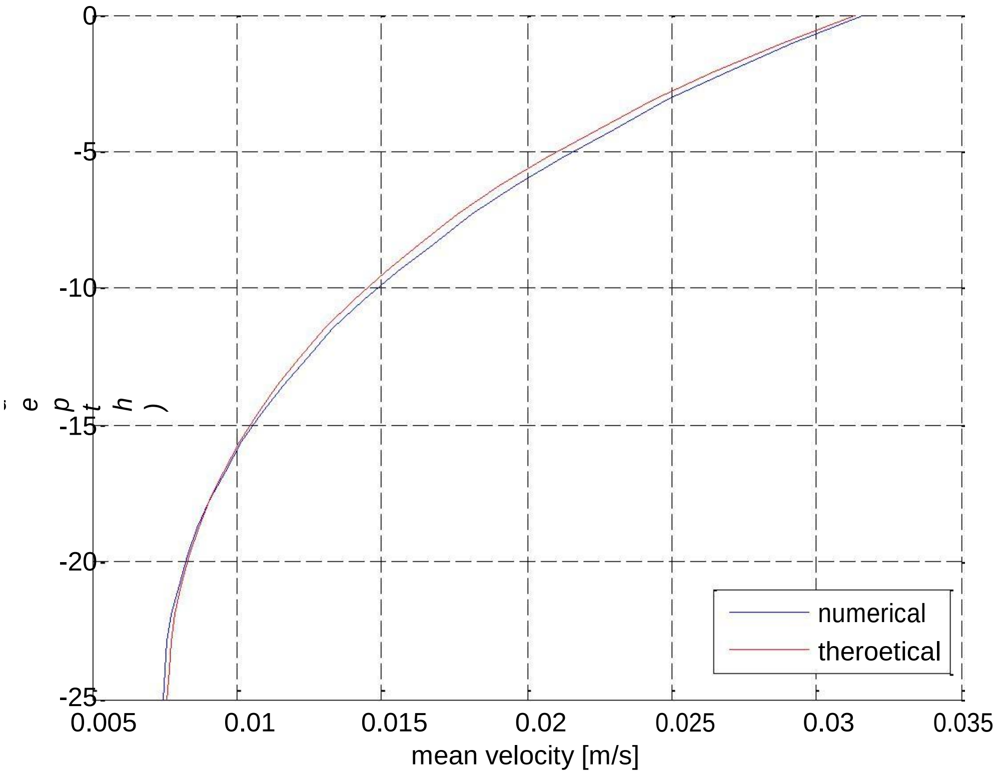

Fig. 12 Mean drift velocity for different depths in the intermediate case

Third case: shallow water

The third case which represents shallow water uses the parameters given on table 4.

On Fig. 13 we can see how the mean drift velocity changes with the depth and that the numerical and the theoretical results are not as close, more error is done. In this case, we see that the velocity decreases as well with the depth but not as much as in the other two cases, in fact in comparison to the first two cases we could say that the mean drift velocity is constant along the depth.

On table 7 we present the mass transport calculated in three ways. Using Eq. 14 (Mt), using the theoretical velocities (Mt_2) and numerically (Mnum). Mnum has been calculated for different number of points (N) along z.

	N=10	N=1000
Mt [kg/s]	0.7245	0.7245
Mt_2 [kg/s]	0.8063	0.7252
Mnum [kg/s]	0. 8036	0. 74320

Table 7

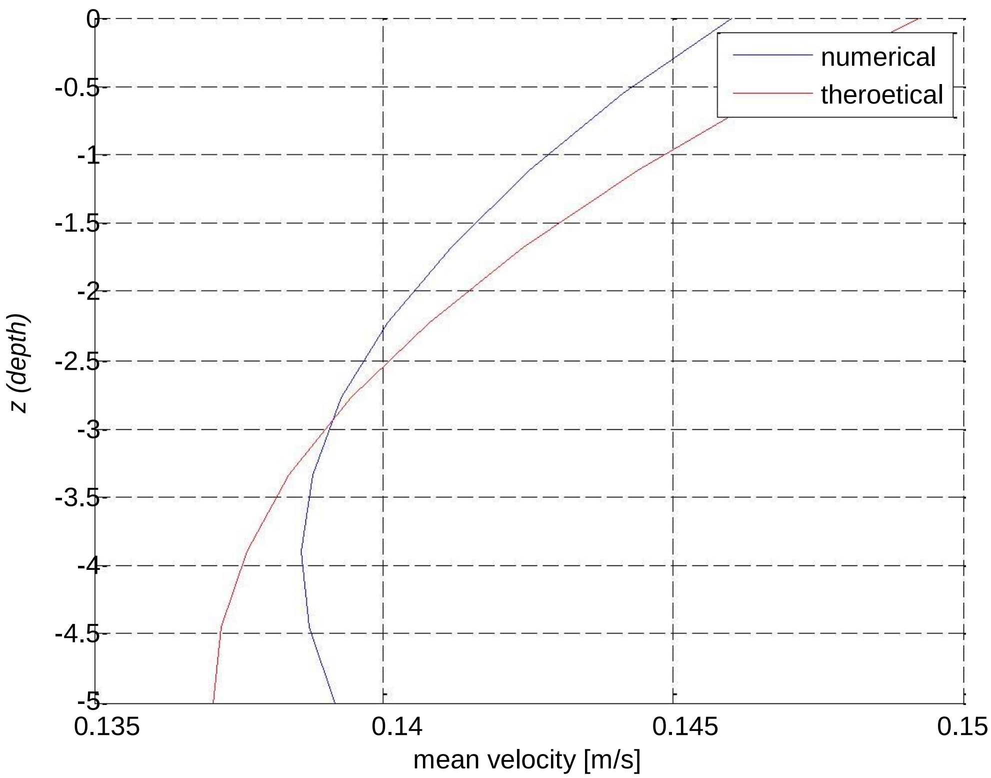

Fig. 13 Mean drift velocity for different depths in shallow water

From the analysis done above we can see that for the same wavelength and amplitude the more water depth we have the less mass transport we have. We have seen that for shallow water we have more mass transport that for deep water. This seems at first strange but if we analysed we can see that this is due to the fact that the period is bigger for shallow water, and therefore is the phase velocity and therefore the mass transport. In fact we can see as well from the graphs above that even if the area per unit of width is less for shallow water the mean drift velocity is much higher than the other two cases.

We have seen as well that for the three cases, when increasing the number of points of discretization along the depth the numerical value for the mass transport is closer to the theoretical value and so do the value calculated with the theoretical velocities. However when number of points along the depth is infinite the value estimated with the theoretical velocities will match the theoretical value, but the value estimated with the numerical velocities will never match the theoretical value as we have seen some error with mean drift velocities.

We have analysed the mean drift velocities as well for different wave amplitudes, from 0.1 to 6m. We have chosen a wavelength of 1000 m a depth of 100 m and T=34 (T is given by the dispersion relation). The results on the theoretical and numerical mean drift velocities on the surface for different amplitudes are presented in Fig. 14. The results for the mass transport are presented on Fig. 15.

From Fig. 14 and Fig. 15 we can see that the mass transport and the mean drift velocity on the surface increase with the amplitude. In fact the relation with the amplitude is parabolic, as we see on the graph and on Eq. 13 and Eq. 14. We also see that for low amplitudes the error is very low and it becomes higher for higher amplitudes, up to 10 % for the mean drift velocity for amplitude of 6 m and less than 1% for the mass transport for amplitude of 6 m. This can be explained that if we only change the amplitude and maintain the wavelength (it is what we have done here), when we increase the amplitude we increase the steepness and therefore the error.

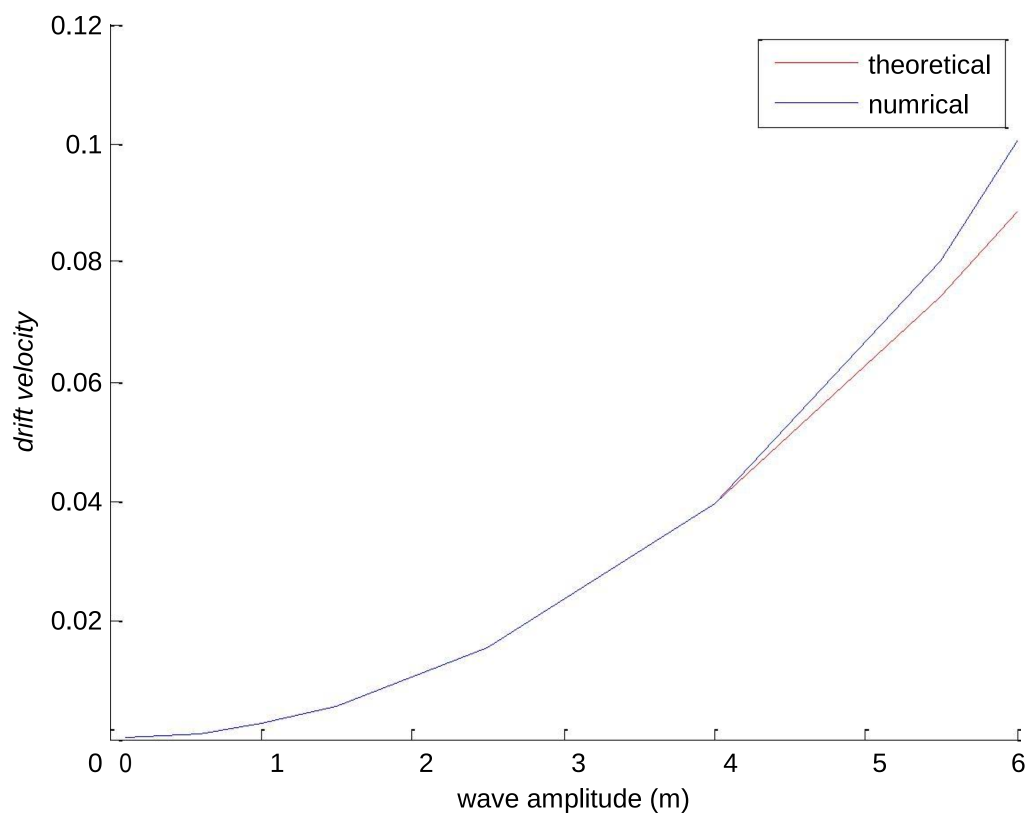

Fig. 14 Mean drift velocity for different wave amplitudes

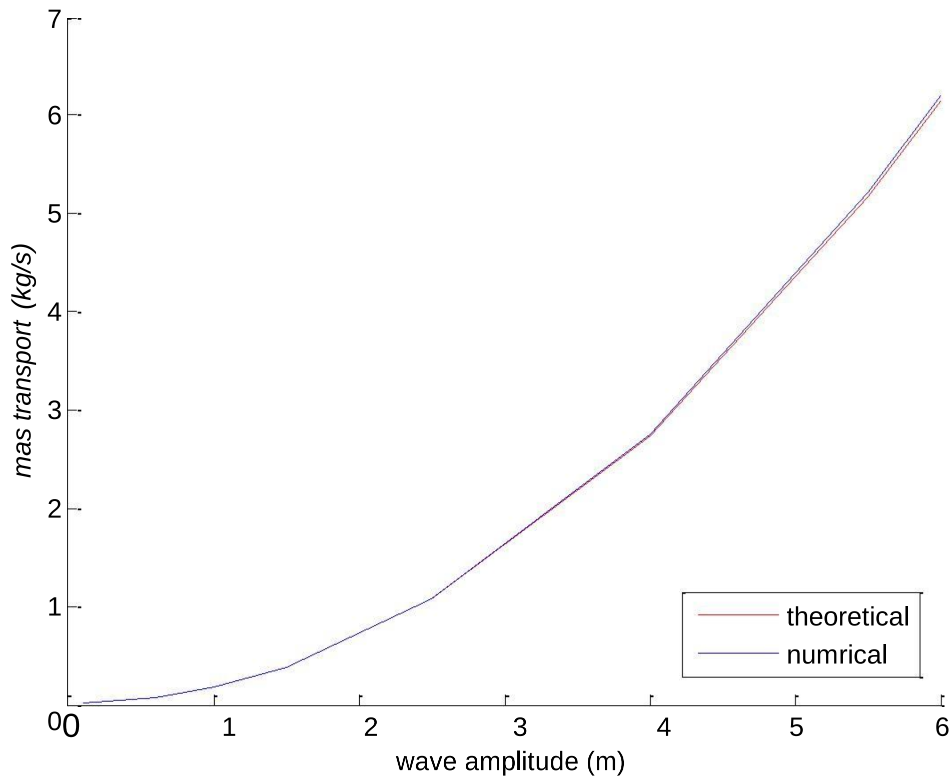

Fig. 15 Mean drift velocity for different depths in shallow water

CONCLUSIONS

For the first part (closed trajectories) we have seen that for deep water the trajectories are almost circular, whereas for the intermediate case and for shallow water the trajectories are ellipses. We have seen as well that for deep water the radius decreases with depth, for an intermediate depth the trajectories are smaller and flatter with depth and for shallow water the trajectories becomes flatted with depth but maintain the width. We can see therefore that the sea bottom has more influence on the shallow water, converting the trajectories more elliptical. The more the water depth the larger are the trajectories as well. We can see that the horizontal displacement of the particle is almost 10 m for shallow water and 1 m for deep water.

For the second part, we have obtained the same results as in the first part but with a drifting motion. The drifting velocity decreases with the depth, and it is more important when the water depth influences more on the particles path (shallow water). This leads us to a higher mass transport for shallow water than for the other cases given that the wavelength and the amplitude is the same for each case. In a real case (working in 2D), due to the geometry of the bottom will have a transition from deep water to shallow water, and the mass flow rate will be the same for the theorem of mass conservation. In order to maintain the mass rate the wave will change its parameters (a, T and λ) depending on the depth.

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