

Multidimensional Boundary-layers for a singularly perturbed Neumann problem

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ABSTRACT. We continue the study of [34], proving concentration phenomena for the equation $-\varepsilon^2 \Delta u + u = u^p$ in a smooth bounded domain $\Omega \subseteq \mathbb{R}^n$ and with Neumann boundary conditions. The exponent p is greater or equal than 1 and the parameter ε is converging to zero. For a suitable sequence $\varepsilon_j \rightarrow 0$ we prove existence of positive solutions u_j concentrating at the whole boundary of Ω or at some of its components.

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1 Introduction

In this paper we are concerned with concentration phenomena for solutions of the singularly-perturbed elliptic problem

$$(P_\varepsilon) \quad \begin{cases} -\varepsilon^2 \Delta u + u = u^p & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega, \end{cases}$$

where $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, is a smooth bounded domain, ν is the interior unit normal vector to $\partial\Omega$, $p > 1$, and ε is a small positive parameter. Our aim is to show the existence of a sequence $\varepsilon_j \rightarrow 0$ for which (P_{ε_j}) admits solutions concentrating at $\partial\Omega$ or at some of its components.

Problem (P_ε) arises in the study of pattern-formation for a class of reaction-diffusion equations. For example, in 1972 Gierer and Meinhardt proposed the following system

$$(GM) \quad \begin{cases} \mathcal{U}_t = d_1 \Delta \mathcal{U} - \mathcal{U} + \frac{\mathcal{U}^p}{\mathcal{V}^q} & \text{in } \Omega \times (0, +\infty), \\ \mathcal{V}_t = d_2 \Delta \mathcal{V} - \mathcal{V} + \frac{\mathcal{U}^r}{\mathcal{V}^s} & \text{in } \Omega \times (0, +\infty), \\ \frac{\partial \mathcal{U}}{\partial \nu} = \frac{\partial \mathcal{V}}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, +\infty), \end{cases}$$

to describe a biological experiment, see [22]. Here the functions $\mathcal{U}$, $\mathcal{V}$ represent the densities of a slowly-diffusing chemical activator and of a fast-diffusing inhibitor respectively. Therefore it is reasonable to assume $d_1 \ll 1 \ll d_2$. The numbers p, q, r, s are non-negative and satisfy the inequalities

$$0 < \frac{p-1}{q} < \frac{r}{s+1}.$$

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By the so-called *Turing's instability*, [43], system (GM) may exhibit non-trivial steady states when the coefficients d_1 and d_2 are very different. In the limit $d_2 \rightarrow +\infty$, such states can be described by solutions of (P_ε) , see the survey [35] for details.

In the case $p < \frac{n+2}{n-2}$, problem (P_ε) admits solutions, called *spike-layers*, concentrating at one or multiple points of $\bar{\Omega}$. It was first established in [32], [36], [37] that least-energy (mountain-pass) solutions of (P_ε) are spike-layers which possess a local maximum point Q_ε at $\partial\Omega$ such that $H(Q_\varepsilon) \rightarrow \max_{\partial\Omega} H$, where H is the mean curvature of $\partial\Omega$.

Further studies on spike-layer solutions (for the Dirichlet problem as well) can be found in [17], [19], [24], [25], [26], [27], [30], [31], [44], [45]. Due to the work of the last ten-fifteen years, the phenomenon of concentration at points is now essentially understood, necessary and sufficient conditions for the location of boundary and interior spikes are available.

Roughly, a spike-layer solution u_ε scales in the following way

$$u_\varepsilon(x) \sim w_0(x/\varepsilon).$$

More precisely, if Q_ε is a local maximum of u_ε , then there holds

$$u_\varepsilon(Q_\varepsilon + \varepsilon \cdot) \rightarrow w_0(\cdot) \quad \text{in } C_{loc}^2(\mathbb{R}^n) \quad (\text{resp. in } C_{loc}^2(\overline{\mathbb{R}_+^n})),$$

where $\mathbb{R}_+^n$ denotes the half-space $\{x_n > 0\}$ and where w_0 is the unique positive radial solution of

$$\begin{cases} -\Delta w_0 + w_0 = w_0^p & \text{in } \mathbb{R}^n \quad (\text{resp. in } \mathbb{R}_+^n); \\ w_0(x) \rightarrow 0 & \text{as } |x| \rightarrow +\infty. \end{cases}$$

The limit domain is $\mathbb{R}^n$ or $\mathbb{R}_+^n$ if Q_ε lies in the interior of Ω or on $\partial\Omega$ respectively.

Spike-layer solutions are produced using minimax methods, gluing techniques and, under suitable non-degeneracy conditions, by finite-dimensional reductions. Very recent results deal with the full system (GM) , see for example [20], [28], [38]. Similar techniques have been used for the study of the Nonlinear Schrödinger Equation (NLS) in the semiclassical limit, or other singularly-perturbed problems, see e.g. [2], [3], [7] and references therein.

It has been conjectured for some time that problem (P_ε) should generically admit solutions concentrating on k -dimensional sets, for $k = 1, \dots, n-1$. The case $k = n$ is excluded since (P_ε) is not expected to exhibit phase transitions.

In presence of symmetry some results have been proved in [4, 5, 6, 8, 10, 18] for problem (P_ε) , the Dirichlet problem, and the semiclassical NLS. See Remark 1.4 below for more precise comments.

In the general case, the above conjecture was first proved in [34] for the case $n = 2$ and $k = 1$, under the technical assumption $p \geq 2$. In the present paper we settle the case $k = n-1$ for all n , assuming only the condition $p > 1$. Hence, besides allowing $p \in (1, 2)$, we can also deal with the supercritical case. This is mainly allowed by the fact that the limit profile of our solutions solves a one-dimensional problem (see the comments below), for which there is no restriction on p . Note that, by the results of [14], in the case of spike-layers (for the stationary NLS equation) the condition $p < \frac{n+2}{n-2}$ is also necessary for concentration.

The solutions we obtain are boundary-layers and scale qualitatively in the following way, as ε tends to zero

$$u_\varepsilon(x', x_n) \sim w_0(x_n/\varepsilon); \quad x' \in \mathbb{R}^{n-1}, x_n \in \mathbb{R}_+,$$

where now w_0 is the solution of the problem

$$(1) \quad \begin{cases} -w_0'' + w_0 = w_0^p & \text{in } \mathbb{R}_+; \\ w_0(x) \rightarrow 0 & \text{as } x \rightarrow +\infty; \\ w_0'(0) = 0. \end{cases}$$

Our main result is the following.

Theorem 1.1 *Let $\Omega \subseteq \mathbb{R}^n$ be a smooth bounded domain, and let $p > 1$. Then there exists a sequence $\varepsilon_j \rightarrow 0$ and a sequence of solutions u_{ε_j} of (P_{ε_j}) with the following properties*

- i) u_{ε_j} concentrates around $\partial\Omega$ as $j \rightarrow +\infty$, namely for every $r > 0$ one has $\int_{\Omega^r} (\varepsilon_j^2 |\nabla u_{\varepsilon_j}|^2 + u_{\varepsilon_j}^2) \rightarrow 0$ as $j \rightarrow +\infty$, where $\Omega^r = \{x \in \Omega : \text{dist}(x, \partial\Omega) \geq r\}$;
- ii) if $x_0 \in \partial\Omega$ and if ν_0 denotes the interior unit normal to $\partial\Omega$ at x_0 , then for every $k \in \mathbb{N}$ there holds

$$u_{\varepsilon_j}(\varepsilon_j(x - x_0)) \rightarrow w_0(x \cdot \nu_0) \quad \text{in } C_{loc}^k(V_0),$$

where $V_0 = \{x \in \mathbb{R}^N : x \cdot \nu_0 > 0\}$, and where w_0 is the solution of (1).

Using the scaling $u(x) \mapsto u(\varepsilon x)$, problem (P_ε) is transformed into

$$(\tilde{P}_\varepsilon) \quad \begin{cases} -\Delta u + u = u^p & \text{in } \Omega_\varepsilon, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega_\varepsilon, \\ u > 0 & \text{in } \Omega_\varepsilon, \end{cases}$$

where $\Omega_\varepsilon = \frac{1}{\varepsilon}\Omega$. For $p \leq \frac{n+2}{n-2}$, solutions of $(\tilde{P}_\varepsilon)$ can be found as critical points of the functional $I_\varepsilon : H^1(\Omega_\varepsilon) \rightarrow \mathbb{R}$ defined by

$$(2) \quad I_\varepsilon(u) = \frac{1}{2} \int_{\Omega_\varepsilon} (|\nabla u|^2 + u^2) - \frac{1}{p+1} \int_{\Omega_\varepsilon} |u|^{p+1}; \quad u \in H^1(\Omega_\varepsilon),$$

where the space $H^1(\Omega_\varepsilon)$ is equipped with its standard norm $\|u\|^2 = \int_{\Omega_\varepsilon} (|\nabla u|^2 + u^2)$. In the following, we use the notation u_ε for both the solutions of (P_ε) and $(\tilde{P}_\varepsilon)$.

The phenomenon of concentration at curves or manifolds for $(\tilde{P}_\varepsilon)$ is very different from that of concentration at points. Essentially, the difficulty here is that the energy of the solutions is very large (of order ε^{1-n}), and so is the Morse Index (also of order ε^{1-n}). The first assertion can be deduced by a simple dimensional argument, while the second is due to a loss of coercivity in the non-scaling direction of u_ε . For an heuristic explanation of this fact, we refer to the introduction of [34], see also Remark 5.2 below. Note also that we can prove existence just along a sequence ε_j , as in [34], or at most along a sequence of intervals in ε , see Remark 5.3. This seems to be a general property of this phenomenon, removable only in special cases, see Remark 1.3.

In the case of spike-layers, both the energy and the Morse index stays bounded as ε goes to zero. Viceversa, by the results in [16], if the Morse index of a family of solutions stays bounded as $\varepsilon \rightarrow 0$, these solutions must concentrate at a finite number of points.

The proof of Theorem 1.1 is based on a local inversion argument. The main difficulty is that, since the Morse index of the solutions is changing with ε , the linearized operator $I_\varepsilon''(u_\varepsilon)$ will not be invertible for all the values of ε in an interval $(0, \varepsilon_0)$. We get invertibility only along a sequence ε_j , and the norm of the inverse operator blows-up at the rate ε_j^{n-1} , see Proposition 4.5. This produces a resonance phenomenon which can be described as follows.

Let $\gamma \in (0, 1)$, and let ν denote the inner unit normal to $\partial\Omega_\varepsilon$. Consider the neighborhood Σ_ε of $\partial\Omega_\varepsilon$ defined as

$$\Sigma_\varepsilon = \{x' + x_n \nu : x' \in \partial\Omega_\varepsilon, x_n \in (0, \varepsilon^{-\gamma})\}.$$

Let $\{\phi_l\}_l$ denote the eigenfunctions of $-\Delta_{\hat{g}}$, where $\Delta_{\hat{g}}$ is the Laplace-Beltrami operator on $\partial\Omega$, and let $\{\lambda_l\}_l$ be the corresponding (non-negative) eigenvalues.

Let z_σ be an eigenfunction of $I_\varepsilon''(u_\varepsilon)$ with eigenvalue σ . Then we can decompose z_σ in Fourier series (in the variable x') in the following way

$$z_\sigma(x', x_n) = \sum_{l=0}^{\infty} \phi_l(\varepsilon x') z_{\sigma,l}(x_n); \quad x' \in \partial\Omega_\varepsilon, x_n \in (0, \varepsilon^{-\gamma}).$$

If the eigenvalue σ is close to zero, it turns out that all the modes $\phi_l z_{\sigma,l}$ are negligible, except those for which $\lambda_l \sim \varepsilon^{-2}$, see also formula (3). This is stated rigorously in Proposition 4.1. It follows that, qualitatively, the resonant eigenvalues of $I''_\varepsilon(u_\varepsilon)$ have eigenfunctions with more and more oscillations along $\partial\Omega_\varepsilon$ as ε tends to zero.

We describe below the general procedure employed here to attack the problem, and the improvements with respect to [34]. The proof of our main result is based on the Contraction Mapping Theorem, once two preliminary steps are accomplished.

Step 1: finding an approximate solution. Given an arbitrary positive number θ , we are able to find an approximate solution $u_{k,\varepsilon}$ of $(\tilde{P}_\varepsilon)$ for which $\|I'_\varepsilon(u_{k,\varepsilon})\| = O(\varepsilon^\theta)$. The function $u_{k,\varepsilon}$ is constructed essentially by power series in ε with k iterations, where the number k depends on n , p and θ . Thus $u_{k,\varepsilon}$ has roughly the following form

$$u_{k,\varepsilon}(x', x_n) = w_0(x_n) + \varepsilon \tilde{w}_1(\varepsilon x', x_n) + \dots \varepsilon^k \tilde{w}_k(\varepsilon x', x_n), \quad x' \in \partial\Omega_\varepsilon, x_n \in J_\varepsilon = [0, \varepsilon^{-\gamma}].$$

Here $\tilde{w}_1, \dots, \tilde{w}_k$ are smooth functions on $\partial\Omega \times J_\varepsilon$, which are defined inductively in their index. Basically, each function $\tilde{w}_i$ is obtained by an inversion argument from $\tilde{w}_0, \dots, \tilde{w}_{i-1}$ and the geometry of Ω . Despite the resonance phenomena of the operator I''_ε , here we can perform this inversion because we are assuming a smooth dependence on the variable $\varepsilon x'$. In some sense, since resonance occurs only at high frequencies, see the comments above, smooth functions on $\partial\Omega$ (scaled to Ω_ε) are not affected by this phenomenon since their Fourier modes are mainly low-frequency ones. The rigorous derivation of $u_{k,\varepsilon}$ is performed in Section 3. The smoothness of Ω is essential to construct $u_{k,\varepsilon}$, see Remark 3.3.

We remark that in [34], for $n = 2$, we were able to satisfy the above requirement only for $\theta = \frac{3}{2}$. Here we need a much better approximation since both the energy of the solutions and the norm of the inverse operator grow faster when the dimension n is larger.

Step 2: inverting the linearized operator $I''_\varepsilon(u_{k,\varepsilon})$. This is a rather delicate issue since, as we remarked before, the linearized operator is not invertible for all the values of ε . Qualitatively, using the Weyl's asymptotic formula, one finds that the l -th eigenvalue σ_l of $I''_\varepsilon(u_{k,\varepsilon})$ is given by

$$(3) \quad \sigma_l \sim -1 + \varepsilon^2 l^{\frac{2}{n-1}},$$

see Section 4 for details. It follows that $\sigma_l \sim 0$ for $l \sim \varepsilon^{1-n}$, and that the average distance between two consecutive eigenvalues close to zero is of order ε^{n-1} . We show indeed that along a sequence ε_j the spectrum of the linearized operator stays away from zero of order ε_j^{n-1} , and hence we get an inverse operator with norm proportional to $\varepsilon_j^{-(n-1)}$. The way to prove this fact is based on a first rough comparison of the eigenvalues with those (essentially known, see Proposition 2.5) of a model problem, obtaining an estimate of the Morse index of the solutions. Then, Kato's Theorem allows us to choose the values of ε appropriately and to invert the linear operator along a sequence ε_j . Notice that Kato's Theorem requires some information not only on the eigenvalues, but also on the eigenfunctions, see Section 2. The procedure here is simpler than [34], where oscillatory integrals were used, see Remark 4.2.

Final step: the contraction argument. If the operator $I''_\varepsilon(u_{k,\varepsilon})$ is invertible, a function u_ε of the form $u_\varepsilon = u_{k,\varepsilon} + w$ is a solution of $(\tilde{P}_\varepsilon)$ if and only if w is a fixed point of the operator F_ε , where F_ε is defined as

$$F_\varepsilon(w) = -I''_\varepsilon(u_{k,\varepsilon})^{-1} [I'_\varepsilon(u_{k,\varepsilon}) + G(w)].$$

Here $G(w)$ is a superlinear term satisfying $\|G(w)\| \leq O(\|w\|)^{\min\{2,p\}}$. Since along the sequence ε_j the norm of the operator $I''_\varepsilon(u_{k,\varepsilon})^{-1}$ is of order ε^{-n} , we need to choose θ to be sufficiently large (see Step 1), depending also on p , in order to get a contraction. Note that the norm of the term $G(w)$ is larger (for w small) if $p \in (1, 2)$, and this case is not treatable with the approach in [34].

A further difficulty in dealing with the case $n \geq 3$ is that the exponent p could also be supercritical. This case is tackled with a truncation argument, proving a priori estimates in L^∞ . These are based on a combination of norm estimates, obtained using Step 2, and elliptic regularity theory.

Remark 1.2 It is possible to generalize Theorem 1.1 in the following way. Let $\Omega \subseteq \mathbb{R}^n$ be a smooth bounded domain, and let $p > 1$. Let $\Phi_1, \dots, \Phi_\ell$ be ℓ components of $\partial\Omega$. Then there exists a sequence $\varepsilon_j \rightarrow 0$ and a sequence of solutions of (P_{ε_j}) which concentrate at $\Phi_1 \cup \dots \cup \Phi_\ell$, namely for which properties i)-ii) in Theorem 1.1 hold for every component $\Phi_1, \dots, \Phi_\ell$. The set Ω_r has to be substituted by $\{x \in \Omega : \text{dist}(x, \Phi_1 \cup \dots \cup \Phi_\ell) \geq r\}$.

Remark 1.3 We do not expect that in general concentration at manifolds of dimension k , with $k \geq 1$, occurs along a continuous range of ε . This is possible (and verified) instead in very particular situations, under some symmetry assumptions. In this case the resonance phenomenon can be ruled out working in spaces of symmetric functions. However, since the Morse index of the solutions is still changing with ε , bifurcation of non-symmetric solutions occurs from the branch of the symmetric ones, see [4], [15], [42].

For the general case it is reasonable to expect imperfect bifurcation, namely that the set $(\varepsilon, u_\varepsilon) \subseteq \mathbb{R} \times H^1(\Omega)$, where u_ε is a boundary-layer, looks like an hyperbola when the Morse index of $u_{k,\varepsilon}$ is changing.

Similar resonance phenomena were observed in [33], [41] for a class of ordinary differential equations. We believe that a similar picture should take place also in this case.

Remark 1.4 Under symmetry assumptions, it is possible to analyze different aspects of concentration at manifolds.

In [4], [8] it is considered the case of standing waves for the semiclassical Nonlinear Schrödinger Equation, and it is shown how the location of the concentration set arises as a combination of the effects of the potential and the volume of the support of the solutions (this phenomenon is absent in the case of concentration at points). In particular in [4] necessary and sufficient conditions are given for the location of the concentration set.

In [8], [18] concentration on circles is produced considering solutions of the NLS with non-vanishing angular momentum, and the potential is now balancing volume and centrifugal forces.

In [5] it is analyzed the effect of the boundary of the domain. In particular, for problem (P_ε) in a ball, it is shown that there exist solutions with a spherical crown of maxima at distance $\varepsilon|\log \varepsilon|$ from the boundary. The radial profile of these solutions is that of an interior spike, hence they are of different type with respect to those analyzed here.

As we already mentioned, in the proof of these results symmetry plays a crucial role. Although these results are very suggestive, there are no analogous examples at the moment dealing with non-symmetric conditions. In particular there is no general result concerning concentration at manifolds with codimension greater than 1. In [4], [5] some open problems and conjectures are listed. It is expected that, if concentration occurs along a d -dimensional manifold, the exponent should be subcritical with respect to $\mathbb{R}^{n-d}$, namely smaller than $\frac{n-d+2}{n-d-2}$. This fact is verified in the present case.

Remark 1.5 Concentration along curves or manifolds also occurs for other classes of equations, like Allen-Cahn, Ginzburg-Landau or Yang-Mills, see e.g. the recent papers [11], [12], [40].

The phenomenon is different from our case, since the resonance phenomena there (although present) do not affect so drastically the structure of the solutions. In fact, under generic non-degeneracy assumptions, concentration occurs without gaps in the scaling parameter.

The main reason for this difference is the following. For these equations the basic solution (transition layer, vortex, instanton) in the direction orthogonal to the manifold of concentration is, loosely speaking a minimum. In our case, instead, the function w_0 is a mountain-pass, giving rise to a negative eigenvalue for the 1-dimensional problem, see Proposition 2.1. As a consequence the eigenvalues of the linearized operator in n dimensions grow from a negative number and eventually cross zero, as described in formula (3).

Remark 1.6 Our approach can also produce solutions with large energy of the following problem on thin annuli

$$\begin{cases} -\Delta u = u^p & \text{in } M_\varepsilon, \\ u = 0 & \text{on } \partial M_\varepsilon \\ u > 0 & \text{in } M_\varepsilon. \end{cases}$$

Here M_ε is an ε -neighborhood of a smooth embedded $(n-1)$ -dimensional manifold $M \subseteq \mathbb{R}^n$, and $p > 1$. Such a problem has been investigated in [9] for $p = \frac{n+2}{n-2}$. The authors proved that there are no solutions whose energy stays bounded as $\varepsilon \rightarrow 0$.

The plan of the paper is the following. In Section 2 we collect some preliminary facts and we study a family of auxiliary one-dimensional problems, proving some continuity and monotonicity properties. Section 3 is devoted to the construction of the approximate solution $u_{k,\varepsilon}$. In Section 4 we give a characterization of the eigenfunctions and the eigenvalues of the operator T_{Σ_ε} , which basically coincide with those of $I_\varepsilon''(u_{k,\varepsilon})$, see Proposition 5.1. In Section 5 we prove our main Theorem, treating separately the subcritical and the supercritical cases.

2 Notation and Preliminary facts

In this section we introduce some notation which will be frequently used later on, and we present the linear theory of some auxiliary one-dimensional problems. Our approach here simplifies that of [34].

Throughout the paper, C denotes a large positive constant. For convenience, we allow C to vary among formulas (also within the same line) and to assume larger and larger values. The number γ will be fixed in Section 3. We often work in the functional space $H^1(\Omega_\varepsilon)$ endowed with its natural norm, which we denote simply by $\|\cdot\|$, without any subscript. Solutions of $(\tilde{P}_\varepsilon)$ for $p \leq \frac{n+2}{n-2}$ are found as critical points of the functional $I_\varepsilon : H^1(\Omega_\varepsilon) \rightarrow \mathbb{R}$ defined in (2).

Consider the problem

$$(4) \quad \begin{cases} -u'' + u = u^p & \text{in } \mathbb{R}_+; \\ u > 0 & \text{in } \mathbb{R}_+; \\ u'(0) = 0, \end{cases}$$

with $p > 1$. It is well-known, see e.g. [21], that problem (4) possesses a solution w_0 which satisfies the properties

$$(5) \quad \begin{cases} w'_0(r) < 0, & \text{for all } r > 0, \\ \lim_{r \rightarrow \infty} e^r w_0(r) = \alpha_p > 0, & \lim_{r \rightarrow \infty} \frac{w'_0(r)}{w_0(r)} = -1, \end{cases}$$

where α_p is a positive constant depending only on p . Using some ODE analysis, one can see that all the solutions of (4) in $H^1(\mathbb{R}_+)$ coincide with w_0 .

Solutions of problem (4) in the class $H^1(\mathbb{R}_+)$ are critical points of the functional $\bar{I}$ defined as

$$(6) \quad \bar{I}(u) = \frac{1}{2} \int_{\mathbb{R}_+} ((u')^2 + u^2) - \frac{1}{p+1} \int_{\mathbb{R}_+} |u|^{p+1}, \quad u \in H^1(\mathbb{R}_+).$$

We have the following non-degeneracy result, see e.g. [21], [39].

Proposition 2.1 *The function w_0 is a non-degenerate critical point of $\bar{I}$. Precisely, there exists a positive constant C such that*

$$\bar{I}''(w_0)[w_0, w_0] \leq -C^{-1} \|w_0\|_0^2; \quad \bar{I}''(w_0)[v, v] \geq C^{-1} \|v\|_0^2, \quad \text{for all } v \in H^1(\mathbb{R}_+), v \perp w_0,$$

where $\|\cdot\|_0$ denotes the standard norm of $H^1(\mathbb{R}_+)$. As a consequence, we have $\mu < 0$ and $\tau > 0$, where μ and τ are respectively the first and the second eigenvalues of $\bar{I}''(w_0)$. Furthermore the eigenvalue μ is simple.

Our next goal is to characterize the eigenvalues σ (in particular the first two) of the following problem

$$(7) \quad \begin{cases} (1 - \sigma)(-u'' + (1 + \alpha)u) = p w_0^{p-1} u, & \text{in } \mathbb{R}_+; \\ u'(0) = 0, \end{cases}$$

where α is a positive parameter. Equation (7) arises in the study of some linear partial differential equation after performing a Fourier decomposition, see equation (26) below. This study has basically been performed in [34] for α ranging in a discrete set of values. Here we present a simpler approach, allowing α to vary continuously.

In order to study the eigenvalues of (7), it is convenient to introduce the following norm $\|\cdot\|_\alpha$ on $H^1(\mathbb{R}_+)$

$$(8) \quad \|u\|_\alpha^2 = \int_{\mathbb{R}_+} ((u')^2 + (1 + \alpha)u^2); \quad u \in H^1(\mathbb{R}_+).$$

Let H_α be the Hilbert space consisting of $H^1(\mathbb{R}_+)$ and endowed with this norm. We denote by $(\cdot, \cdot)_\alpha$ the corresponding scalar product. We also define by duality the operator $T_\alpha : H_\alpha \rightarrow H_\alpha$ in the following way

$$(T_\alpha u, v)_\alpha = \int_{\mathbb{R}_+} (u'v' + (1 + \alpha)uv - pw_0^{p-1}uv); \quad u, v \in H_\alpha.$$

Then equation (7) can be written in the abstract form

$$(9) \quad T_\alpha u = \sigma u; \quad u \in H_\alpha.$$

Note that when $\alpha = 0$, T_α coincides with $\bar{T}''(w_0)$. We have the following result.

Proposition 2.2 *Let μ_α and τ_α denote respectively the first and the second eigenvalues of T_α . Then μ_α is simple and the following properties hold true*

- (i) $\alpha \mapsto \mu_\alpha$ is smooth and strictly increasing;
- (ii) $\alpha \mapsto \tau_\alpha$ is non-decreasing.

The eigenfunction v_α of T_α corresponding to μ_α , normalized with $\|v_\alpha\|_\alpha = 1$, can be chosen to be positive and strictly decreasing on $\mathbb{R}_+$. The map $\alpha \mapsto v_\alpha$ is smooth from $\mathbb{R}$ into $H^1(\mathbb{R}_+)$. Furthermore, $\mu_\alpha > 0$ for α sufficiently large.

PROOF. The simplicity on μ_α and the (weak) monotonicity of $\alpha \mapsto \mu_\alpha$, $\alpha \mapsto \tau_\alpha$ can be proved as in [34], Section 3.

The smoothness of $\alpha \mapsto \mu_\alpha$ and $\alpha \mapsto v_\alpha$ can be deduced as follows. Consider the map $\Psi_\alpha : H^1(\mathbb{R}_+) \rightarrow H_\alpha$ defined as

$$(\Psi_\alpha u)(x) = u(x); \quad u \in H^1(\mathbb{R}_+).$$

Ψ_α is nothing but the identity as a map between functions, and is an isomorphism between Sobolev spaces. We also consider the operator $\mathcal{T}_\alpha : H^1(\mathbb{R}_+) \rightarrow H^1(\mathbb{R}_+)$

$$\mathcal{T}_\alpha = \Psi_\alpha^{-1} \circ T_\alpha \circ \Psi_\alpha.$$

$\mathcal{T}_\alpha$ depends smoothly on α and, by conjugacy to T_α , the first eigenvalue of $\mathcal{T}_\alpha$ coincides with μ_α and is simple. Hence, to get the smoothness of μ_α and v_α , it is sufficient to apply the Implicit Function Theorem.

We now compute the derivative of μ_α with respect to α . The function v_α satisfies the equation

$$(10) \quad \begin{cases} (1 - \mu_\alpha)(-v_\alpha'' + (1 + \alpha)v_\alpha) = pw_0^{p-1}v_\alpha & \text{in } \mathbb{R}_+; \\ v_\alpha'(0) = 0. \end{cases}$$

Differentiating the equation $\|v_\alpha\|_\alpha^2 = 1$, with respect to α , we find

$$(11) \quad \frac{d}{d\alpha} \|v_\alpha\|_\alpha^2 = 0 \quad \text{which implies} \quad \left(\frac{dv_\alpha}{d\alpha}, v_\alpha \right)_\alpha = - \int_{\mathbb{R}_+} v_\alpha^2.$$

On the other hand, differentiating (10) with respect to α , we obtain

$$(12) \quad \begin{cases} -\frac{d\mu_\alpha}{d\alpha} (-v''_\alpha + (1+\alpha)v_\alpha) + (1-\mu_\alpha) \left(-\left(\frac{dv_\alpha}{d\alpha}\right)'' + (1+\alpha)\frac{dv_\alpha}{d\alpha} + v_\alpha \right) = pw_0^{p-1} \frac{dv_\alpha}{d\alpha} & \text{in } \mathbb{R}_+; \\ \left(\frac{dv_\alpha}{d\alpha}\right)'(0) = 0. \end{cases}$$

Multiplying (12) by v_α , integrating by parts and using (11), one gets

$$(13) \quad \frac{d\mu_\alpha}{d\alpha} = (1-\mu_\alpha) \int_{\mathbb{R}_+} v_\alpha^2 > 0.$$

Note that, since $T_\alpha \leq Id$, every eigenvalue of T_α is strictly less than 1, and in particular $(1-\mu_\alpha) > 0$. This proves the strict monotonicity of $\alpha \mapsto \mu_\alpha$. It remains to prove that $\mu_\alpha > 0$ for α large: in order to do this we show that $\mu_\alpha \rightarrow 1$ as $\alpha \rightarrow +\infty$. There holds

$$\mu_\alpha = \inf_{u \in H_\alpha} \frac{\int_{\mathbb{R}_+} \left((u')^2 + (1+\alpha)u^2 - pw_0^{p-1}u^2 \right)}{\int_{\mathbb{R}_+} ((u')^2 + (1+\alpha)u^2)}.$$

Fixing any $\delta > 0$, it is sufficient to notice that

$$(u')^2 + \left((1+\alpha) - pw_0^{p-1} \right) u^2 \geq (1-\delta) [(u')^2 + (1+\alpha)u^2] \quad \text{for all } u,$$

provided α is sufficiently large. This concludes the proof. ■

Given $\alpha > 0$, we also define the function $\tilde{F}(\alpha)$ to be

$$(14) \quad \tilde{F}(\alpha) = 2\alpha(1-\mu_\alpha) \int_{\mathbb{R}_+} v_\alpha^2 > 0.$$

Note that, by the smoothness of $\alpha \mapsto v_\alpha$, also the function $\alpha \mapsto \tilde{F}(\alpha)$ is smooth.

We also need to consider a variant of the eigenvalue problem (7). For $\gamma > 0$, let J_ε denote the interval

$$J_\varepsilon = [0, \varepsilon^{-\gamma}],$$

and let

$$H_\varepsilon^1 = \{u \in H^1(J_\varepsilon) : u(\varepsilon^{-\gamma}) = 0\}.$$

We let $H_{\alpha,\varepsilon}$ denote the space H_ε^1 endowed with the norm

$$\|u\|_{\alpha,\varepsilon}^2 = \int_{J_\varepsilon} ((u')^2 + (1+\alpha)u^2); \quad u \in H_\varepsilon^1,$$

and $(\cdot, \cdot)_{\alpha,\varepsilon}$ the corresponding scalar product. Similarly, we define $T_{\alpha,\varepsilon}$ by

$$(T_{\alpha,\varepsilon}u, v)_{\alpha,\varepsilon} = \int_{J_\varepsilon} \left(u'v' + (1+\alpha)uv - pw_0^{p-1}uv \right); \quad u, v \in H_{\alpha,\varepsilon}.$$

The operator $T_{\alpha,\varepsilon}$ satisfies properties analogous to T_α . We list them in the next Proposition, which also gives a comparison between the first eigenvalues and eigenfunctions of T_α and $T_{\alpha,\varepsilon}$.

Proposition 2.3 *Let $\mu_{\alpha,\varepsilon}$ and $\tau_{\alpha,\varepsilon}$ denote respectively the first and the second eigenvalues of $T_{\alpha,\varepsilon}$. Then $\mu_{\alpha,\varepsilon}$ is simple and the following properties hold true*

- (i) $\alpha \mapsto \mu_{\alpha,\varepsilon}$ is smooth and strictly increasing;
- (ii) $\alpha \mapsto \tau_{\alpha,\varepsilon}$ is non-decreasing.

The eigenfunction $v_{\alpha,\varepsilon}$ of $T_{\alpha,\varepsilon}$ corresponding to $\mu_{\alpha,\varepsilon}$, normalized with $\|v_{\alpha,\varepsilon}\|_{\alpha,\varepsilon} = 1$, can be chosen to be positive and strictly decreasing on $\mathbb{R}_+$. Moreover there exist a large constant C and a small constant δ depending only on p such that

$$|\mu_\alpha - \mu_{\alpha,\varepsilon}| \leq C e^{-\delta \varepsilon^{-\gamma}}; \quad \|v_\alpha - v_{\alpha,\varepsilon}\|_{H^1(\mathbb{R}_+)} \leq C \varepsilon^{-\frac{1}{2}\gamma} e^{-\delta \varepsilon^{-\gamma}}, \quad \text{for } \mu_\alpha \in \left[-\frac{1}{4}\tau, \frac{1}{4}\tau\right]$$

and for ε small. The function $v_{\alpha,\varepsilon}$ in this formula has been set identically 0 outside $[0, \varepsilon^{-\gamma}]$.

PROOF. Properties (i)-(ii) and of the monotonicity of $v_{\alpha,\varepsilon}$ can be deduced as for Proposition 2.2.

There holds

$$(15) \quad \mu_\alpha = \inf_{u \in H_\alpha} \frac{(T_\alpha u, u)_\alpha}{\|u\|_\alpha^2}; \quad \mu_{\alpha,\varepsilon} = \inf_{u \in H_{\alpha,\varepsilon}} \frac{(T_{\alpha,\varepsilon} u, u)_{\alpha,\varepsilon}}{\|u\|_{\alpha,\varepsilon}^2}.$$

Since the norms $\|\cdot\|_\alpha$ and $\|\cdot\|_{\alpha,\varepsilon}$ (and the operators $T_\alpha, T_{\alpha,\varepsilon}$) coincide on $H_{\alpha,\varepsilon}$, we have $\mu_\alpha \leq \mu_{\alpha,\varepsilon}$. On the other hand, since $\mu_\alpha \rightarrow 1$ as $\alpha \rightarrow +\infty$ (see the proof of Proposition 2.2), $\mu_\alpha \in [-\frac{1}{4}\tau, \frac{1}{4}\tau]$ implies that α lies in a bounded set of $\mathbb{R}_+$. Note that, since $\bar{I}''(w_0) \leq Id$ on $H^1(\mathbb{R}_+)$, τ is strictly less than 1, and so is $\frac{\tau}{4}$.

Then, using equation (7), we deduce the following decay for v_α

$$(16) \quad |v_\alpha(t)| + |v'_\alpha(t)| \leq C e^{-\delta t}, \quad t \geq 0.$$

Let us define a non-increasing cut-off function φ_ε satisfying the following properties

$$(17) \quad \begin{cases} \varphi_\varepsilon(t) = 1 & \text{for } t \in [0, \frac{1}{2}\varepsilon^{-\gamma}]; \\ \varphi_\varepsilon(t) = 0 & \text{for } t \in [\frac{3}{4}\varepsilon^{-\gamma}, \varepsilon^{-\gamma}]; \\ |\varphi'_\varepsilon(t)| \leq C\varepsilon^\gamma; \quad |\varphi''_\varepsilon(t)| \leq C\varepsilon^{2\gamma}. \end{cases}$$

Then, using (16) and elementary computations, one finds

$$\| \varphi_\varepsilon v_\alpha \|_{\alpha,\varepsilon}^2 - 1 \leq C e^{-\delta \varepsilon^{-\gamma}}; \quad |(T_{\alpha,\varepsilon}(\varphi_\varepsilon v_\alpha), \varphi_\varepsilon v_\alpha) - \mu_\alpha| \leq C e^{-\delta \varepsilon^{-\gamma}}.$$

Since $\varphi_\varepsilon v_\alpha$ belongs to $H_{\alpha,\varepsilon}$, (15) and the last formulas yield $\mu_{\alpha,\varepsilon} \leq \mu_\alpha + C e^{-\delta \varepsilon^{-\gamma}}$. This proves the first inequality in the statement.

Formula (16) implies

$$\int_{\mathbb{R}_+ \setminus J_\varepsilon} ((v'_\alpha)^2 + v_\alpha^2) \leq C \varepsilon^{-\delta \varepsilon^{-\gamma}}.$$

To obtain the second inequality in the statement we set

$$u(t) = v_{\alpha,\varepsilon}(t) - v_\alpha(t) + \varepsilon^{2\gamma} v_\alpha(\varepsilon^{-\gamma} t^2); \quad t \in J_\varepsilon.$$

Subtracting the differential equations satisfied by $v_{\alpha,\varepsilon}$ and v_α , we obtain

$$(18) \quad \begin{cases} (1 - \mu_{\alpha,\varepsilon})(-u'' + (1 + \alpha)u) - p w_0^{p-1} u = f_1 + f_2 & \text{in } J_\varepsilon, \\ u'(0) = 0 & u \in H_{\alpha,\varepsilon}, \end{cases}$$

where f_1 and f_2 are defined by duality as

$$(f_1, v) = (\mu_{\alpha,\varepsilon} - \mu_\alpha) \int_{J_\varepsilon} (-\Delta v_\alpha + (1 + \alpha)v_\alpha) v dt, \quad \forall v \in H_{\alpha,\varepsilon};$$

$$(f_2, v) = \varepsilon^{2\gamma} v_\alpha(\varepsilon^{-\gamma}) \int_{J_\varepsilon} [(1 - \mu_{\alpha,\varepsilon})((1 + \alpha)t^2 - 2) - p w_0^{p-1} t^2] v dt, \quad \forall v \in H_{\alpha,\varepsilon}.$$

one can check that

$$(19) \quad \|f_1\|_{H_{\alpha,\varepsilon}} \leq C e^{-\delta\varepsilon^{-\gamma}}; \quad \|f_2\|_{H_{\alpha,\varepsilon}} \leq C \varepsilon^{-\frac{1}{2}\gamma} e^{-\delta\varepsilon^{-\gamma}}.$$

We write

$$(20) \quad u = \beta v_{\alpha,\varepsilon} + \bar{u}, \quad \text{where} \quad \beta \in \mathbb{R} \text{ and } (\bar{u}, v_{\alpha,\varepsilon})_{H_{\alpha,\varepsilon}} = 0.$$

Since $T_{\alpha,\varepsilon}$ is invertible on $(v_{\alpha,\varepsilon})^\perp$, (19) implies

$$(21) \quad \|\bar{u}\|_{H_{\alpha,\varepsilon}} \leq C \varepsilon^{-\frac{1}{2}\gamma} e^{-\delta\varepsilon^{-\gamma}}.$$

Equation (20) can also be written as

$$(1 - \beta)v_{\alpha,\varepsilon} = \bar{u} + (v_\alpha - \varepsilon^{2\gamma} v_\alpha(\varepsilon^{-\gamma})t^2).$$

Taking the norm of both sides, using (21) and some elementary estimates we get

$$|1 - \beta| = 1 + O\left(\varepsilon^{-\frac{1}{2}\gamma} e^{-\delta\varepsilon^{-\gamma}}\right).$$

This equation implies $\beta \sim 0$ or $\beta \sim 2$, from which we deduce (see (20)) that either $v_{\alpha,\varepsilon} \sim v_\alpha$, or $v_{\alpha,\varepsilon} \sim -v_\alpha$. Since we are assuming that $v_{\alpha,\varepsilon}$ is non-increasing, it must be $\beta \sim 0$, and more precisely $|\beta| \leq C \varepsilon^{-\frac{1}{2}\gamma} e^{-\delta\varepsilon^{-\gamma}}$. Then (20) and (21) yield the conclusion. ■

Remark 2.4 Reasoning as in the proof of Proposition 2.3 and using the Courant-Fisher method, one can also prove that $\tau_{\alpha,\varepsilon} \geq \tau_\alpha$ for all values of α and ε .

As in [34], we will use Fourier analysis in order to study equation $(\tilde{P}_\varepsilon)$. For $\gamma > 0$, let us define the set S_ε and the metric g_0 on S_ε as

$$(22) \quad S_\varepsilon = \partial\Omega_\varepsilon \times J_\varepsilon = \partial\Omega_\varepsilon \times [0, \varepsilon^{-\gamma}], \quad g_0 = \bar{g}_\varepsilon \otimes (dt)^2,$$

where $\bar{g}_\varepsilon$ is the metric on $\partial\Omega_\varepsilon$ induced by $\mathbb{R}^n$. We also define the functional space

$$H_{S_\varepsilon} = \{u \in H^1(S_\varepsilon) : u(x', \varepsilon^{-\gamma}) = 0 \text{ for all } x' \in \partial\Omega_\varepsilon\}$$

endowed with the norm

$$\|u\|_{H_{S_\varepsilon}}^2 = \int_{S_\varepsilon} (|\nabla_{g_0} u|^2 + u^2); \quad u \in H_{S_\varepsilon}.$$

Let $\hat{g}$ be the metric on $\partial\Omega$ induced by that of $\mathbb{R}^n$, and let $\Delta_{\hat{g}}$ be the Laplace-Beltrami operator. Let $\{\lambda_l\}_l$ denote the eigenvalues of $-\Delta_{\hat{g}}$, counted in non-decreasing order and with their multiplicity. Let also $\{\phi_l\}_l$ denote the corresponding eigenfunctions.

Given $u \in H_{S_\varepsilon}$, we can decompose it in Fourier components (in the variable x') as follows

$$(23) \quad u(x', x_n) = \sum_{l=0}^{\infty} \phi_l(\varepsilon x') u_l(x_n), \quad x' \in \partial\Omega_\varepsilon, x_n \in J_\varepsilon.$$

Using this decomposition, one finds

$$\|u\|_{H_{S_\varepsilon}}^2 = \frac{1}{\varepsilon^{n-1}} \sum_l \int_{J_\varepsilon} ((u'_l)^2 + (1 + \varepsilon^2 \lambda_l) u_l^2) = \frac{1}{\varepsilon^{n-1}} \sum_l \|u_l\|_{\varepsilon^2 \mu_l}^2.$$

Writing for brevity $\|\cdot\|_{l,\varepsilon}$ instead of $\|\cdot\|_{\varepsilon^2 \mu_l}$, the last equation becomes

$$(24) \quad \|u\|_{H_{S_\varepsilon}}^2 = \frac{1}{\varepsilon^{n-1}} \sum_l \|u_l\|_{l,\varepsilon}^2.$$

This is how the norms $\|\cdot\|_\alpha$ introduced above enter in our study; in particular we will choose α belonging to the discrete set $\{\varepsilon^2 \lambda_l\}_l$.

We are also interested in an eigenvalue problem of the form

$$(25) \quad T_{S_\varepsilon} u = \sigma u; \quad u \in H_{S_\varepsilon},$$

where T_{S_ε} is defined by

$$(T_{S_\varepsilon} u, v) = \int_{S_\varepsilon} \left(\nabla_{g_0} u \cdot \nabla_{g_0} v + uv - pw_0^{p-1} uv \right) dV_{g_0}, \quad u, v \in H_{S_\varepsilon}.$$

Writing u as in (23), equation (25) is equivalent to

$$(26) \quad \begin{cases} (1 - \sigma) (-u_l'' + (1 + \varepsilon^2 \lambda_l) u_l) = pw_0^{p-1} u_l & \text{in } J_\varepsilon, \\ u_l'(0) = 0; \end{cases} \quad \text{for all } l,$$

which is of the form (10). since T_{S_ε} represents a model for the study of $I_\varepsilon''(u_{k,\varepsilon})$, it was essential for us to perform a spectral analysis of the operators T_α .

The spectrum of T_{S_ε} is characterized in the next proposition. We recall the definition of the (positive) number τ in Proposition 2.1.

Proposition 2.5 *Let $\sigma < \frac{\tau}{4}$ be an eigenvalue of T_{S_ε} . Then $\sigma = \mu_{l,\varepsilon}$ for some index l . The corresponding eigenfunctions u are of the form*

$$(27) \quad u(x', x_n) = \sum_{\{l : \mu_{l,\varepsilon} = \sigma\}} \alpha_l \phi_l(\varepsilon x') v_{l,\varepsilon}(x_n), \quad x' \in \partial\Omega_\varepsilon, x_n \in J_\varepsilon,$$

where $\{\alpha_l\}_l$ are arbitrary constants. Viceversa, every function of the form (27) is an eigenfunction of T_{S_ε} with eigenvalue σ . In particular the eigenvalues of T_{S_ε} which are smaller than τ coincide with the numbers $\{\mu_{l,\varepsilon}\}_l$ which are smaller than τ .

PROOF. Let u be an eigenfunction of T_{S_ε} with eigenvalue σ . Then we can write

$$u(x', x_n) = \sum_l (\alpha_l v_{l,\varepsilon}(x_n) + \hat{v}_l(x_n)) \phi_l(\varepsilon x'),$$

where α_l are real numbers, and where $(\hat{v}_l, v_{l,\varepsilon})_{l,\varepsilon} = 0$ for all l . We have

$$\sum_l (\alpha_l \mu_{l,\varepsilon} v_{l,\varepsilon}(x_n) + T_{l,\varepsilon} \hat{v}_l(x_n)) \phi_l(\varepsilon x') = T_{S_\varepsilon} u = \sigma u = \sigma \sum_l (\alpha_l v_{l,\varepsilon}(x_n) + \hat{v}_l(x_n)) \phi_l(\varepsilon x').$$

By the uniqueness of the Fourier decomposition, the last equation implies

$$(28) \quad \mu_{l,\varepsilon} = \sigma \quad \text{if} \quad \alpha_l \neq 0; \quad \text{and} \quad T_{l,\varepsilon} \hat{v}_l = \sigma \hat{v}_l \quad \text{for all } l.$$

Proposition 2.3 ii) and Remark 2.4 yield

$$(29) \quad \tau_{\alpha,\varepsilon} \geq \tau_\alpha \geq \tau,$$

which means that $T_{\alpha,\varepsilon} \geq \tau Id$ on the subspace of $H_{\alpha,\varepsilon}$ orthogonal to $v_{\alpha,\varepsilon}$. Hence the second equation in (28) and the fact that $\sigma < \frac{\tau}{4}$ imply $\hat{v}_l = 0$ for all l . Moreover, since $\lambda_l \rightarrow \mu_{l,\varepsilon}$ is a monotone function, the first equation in (29) shows that all the indices l for which $\alpha_l \neq 0$ correspond to the same value λ_l . This concludes the proof. ■

We finally recall the following theorem due to T. Kato, ([29], page 444) which is fundamental to us in order to obtain invertibility, see Proposition 4.5.

Theorem 2.6 *Let $T(\chi)$ denote a differentiable family of operators from an Hilbert space X into itself, where χ belongs to an interval containing 0. Let $T(0)$ be a self-adjoint operator of the form Identity - compact and let $\sigma(0) = \sigma_0 \neq 1$ be an eigenvalue of $T(0)$. Then the eigenvalue $\sigma(\chi)$ is differentiable at 0 with respect to χ . The derivative of σ is given by*

$$\frac{\partial \sigma}{\partial \chi} = \left\{ \text{eigenvalues of } P_{\sigma_0} \circ \frac{\partial T}{\partial \chi}(0) \circ P_{\sigma_0} \right\},$$

where $P_{\sigma_0} : X \rightarrow X_{\sigma_0}$ denotes the projection onto the σ_0 eigenspace X_{σ_0} of $T(0)$.

Remark 2.7 *We note that, when perturbing the operator $T(0)$, the eigenvalue σ_0 can split in several ones. Hence in general σ_0 possesses a multivalued derivative. Anyway, since the operator $T(0)$ is of the form Identity - compact, and since σ_0 is different from 1, σ_0 is an isolated eigenvalue and the projection P_{σ_0} has a finite dimensional range. Hence the splitting of σ_0 is finite.*

3 An approximate solution

In this section we construct an approximate solution $u_{k,\varepsilon}$ of $(\tilde{P}_\varepsilon)$. Differently from [34], we find an approximation at arbitrary order in ε . Basically we expand a solution u_ε of $(\tilde{P}_\varepsilon)$ in powers of ε and then use suitable truncations.

Proposition 3.1 *Consider the Euler functional I_ε defined in (2) and associated to problem $(\tilde{P}_\varepsilon)$. Then for any $k \in \mathbb{N}$ there exists a function $u_{k,\varepsilon} : S_\varepsilon \rightarrow \mathbb{R}$ with the following properties*

$$(30) \quad \|I'_\varepsilon(u_{k,\varepsilon})\| \leq C_k \varepsilon^{k+1-\frac{n-1}{2}}; \quad u_{k,\varepsilon} \geq 0 \quad \text{in } \Omega_\varepsilon \quad \frac{\partial u_{k,\varepsilon}}{\partial \nu} = 0 \quad \text{on } \partial\Omega_\varepsilon,$$

where C_k depends only on Ω , p and k . Moreover there holds

$$(31) \quad \begin{cases} \|(\nabla')^{(m)} u_{k,\varepsilon}(x', x_n)\|' \leq C_m \varepsilon^m e^{-x_n}; \\ \|(\nabla')^{(m)} \partial_{x_n} u_{k,\varepsilon}(x', x_n)\|' \leq C_m \varepsilon^m e^{-x_n}; \\ \|(\nabla')^{(m)} \partial_{x_n}^2 u_{k,\varepsilon}(x', x_n)\|' \leq C_m \varepsilon^m e^{-x_n}; \end{cases} \quad x' \in \partial\Omega_\varepsilon, x_n \in J_\varepsilon, m = 0, 1, \dots,$$

where ∇' and $\|\cdot\|'$ denote the derivative and the norm taken with respect to the variable x' (freezing x_n), and C_m is a constant depending only on Ω , p and m .

We parameterize the set S_ε , see equation (22), using coordinates x' on $\partial\Omega_\varepsilon$ and x_n in J_ε . Let ν denote the unit inner normal to $\partial\Omega$, and define the map $\Gamma_\varepsilon : \partial\Omega_\varepsilon \times J_\varepsilon \rightarrow \mathbb{R}^n$ by

$$\Gamma_\varepsilon(x', x_n) = x' + x_n \nu(\varepsilon x').$$

We let the upper-case indices $A, B, C, \dots$ run from 1 to n , and the lower-case indices $i, j, k, \dots$ run from 1 to $n-1$. Using some local coordinates $\{x_i\}_{i=1, \dots, n-1}$ on $\partial\Omega_\varepsilon$, and letting φ_ε be the corresponding immersion into $\mathbb{R}^n$, we have

$$\begin{cases} \frac{\partial \Gamma_\varepsilon}{\partial x_i}(x', x_n) = \frac{\partial \varphi_\varepsilon}{\partial x_i}(x') + \varepsilon x_n \frac{\partial \nu}{\partial x_i}(\varepsilon x') = \frac{\partial \varphi_\varepsilon}{\partial x_i}(x') + \varepsilon x_n H_i^j(\varepsilon x') \frac{\partial \varphi_\varepsilon}{\partial x_j}(x'), & \text{for } i = 1, \dots, n-1; \\ \frac{\partial \Gamma_\varepsilon}{\partial x_n}(x', x_n) = \nu(\varepsilon x'). \end{cases}$$

where $\{H_i^j\}$ are the coefficients of the mean-curvature operator on $\partial\Omega$. Let also $\bar{g}_{ij}$ be the coefficients of the metric on $\partial\Omega_\varepsilon$ in the above coordinates (x', x_n) . Then, letting $g = g_\varepsilon$ denote the metric on Ω_ε induced by $\mathbb{R}^n$, we have

$$(32) \quad g_{AB} = \left(\frac{\partial \Gamma_\varepsilon}{\partial x_A}, \frac{\partial \Gamma_\varepsilon}{\partial x_B} \right) = \begin{pmatrix} \{g_{ij}\} & 0 \\ 0 & 1 \end{pmatrix},$$

where

$$\begin{aligned} g_{ij} &= \left(\frac{\partial \varphi_\varepsilon}{\partial x_i}(x') + \varepsilon x_n H_i^k(\varepsilon x') \frac{\partial \varphi_\varepsilon}{\partial x_k}(x'), \frac{\partial \varphi_\varepsilon}{\partial x_j}(x') + \varepsilon x_n H_j^l(\varepsilon x') \frac{\partial \varphi_\varepsilon}{\partial x_l}(x') \right) \\ &= \bar{g}_{ij} + \varepsilon x_n (H_i^k \bar{g}_{kj} + H_j^l \bar{g}_{il}) + \varepsilon^2 x_n^2 H_i^k H_j^l \bar{g}_{kl}. \end{aligned}$$

Note that also the inverse matrix $\{g^{AB}\}$ decomposes as

$$g^{AB} = \begin{pmatrix} \{g^{ij}\} & 0 \\ 0 & 1 \end{pmatrix}.$$

3.1 First-order approximation

We begin by finding a first-order approximation for u_ε , to show the ideas of the general procedure. The approach of this subsection is simpler than [34], although closely related. We define the following map $u \mapsto \tilde{u}$ from functions on $\partial\Omega_\varepsilon$ (resp. $\partial\Omega_\varepsilon \times \mathbb{R}_+$) into functions on $\partial\Omega$ (resp. $\partial\Omega \times \mathbb{R}_+$)

$$(33) \quad \tilde{u}(\varepsilon x') = u(x') \quad (\text{resp. } \tilde{u}(\varepsilon x', x_n) = u(x', x_n)), \quad x' \in \partial\Omega_\varepsilon \quad (\text{resp. } (x', x_n) \in \partial\Omega_\varepsilon \times \mathbb{R}_+).$$

Using the above parametrization, we look for an approximate solution $u_{1,\varepsilon}$ of the form

$$u_{1,\varepsilon}(x', x_n) = w_0 + \varepsilon \tilde{w}_1(\varepsilon x', x_n),$$

where the function $\tilde{w}_1$ is defined on $\partial\Omega \times \mathbb{R}_+$, consistently with (33). We note that, from the above decomposition of g_{AB} (and g^{AB}), one has

$$\begin{aligned} -\Delta_g u &= -g^{AB} u_{AB} - \frac{1}{\sqrt{\det g}} \partial_A \left(g^{AB} \sqrt{\det g} \right) u_B \\ &= -u_{nn} - g^{ij} u_{ij} - \frac{1}{\sqrt{\det g}} \partial_n \left(\sqrt{\det g} \right) u_n - \frac{1}{\sqrt{\det g}} \partial_i \left(g^{ij} \sqrt{\det g} \right) u_j. \end{aligned}$$

We have, formally

$$(34) \quad \det g = \det(\bar{g}^{-1} g) \det \bar{g} = (\det \bar{g}) (1 + \varepsilon x_n \operatorname{tr}(\bar{g}^{-1} \alpha)) + o(\varepsilon),$$

where

$$\alpha_{ij} = H_i^k \bar{g}_{kj} + H_j^l \bar{g}_{il}.$$

There holds

$$(\bar{g}^{-1} \alpha)_{is} = \bar{g}^{sj} \alpha_{ij} = \bar{g}^{sj} (H_i^k \bar{g}_{kj} + H_j^l \bar{g}_{il}),$$

and hence

$$(35) \quad \operatorname{tr}(\bar{g}^{-1} \alpha) = \bar{g}^{ij} (H_i^k \bar{g}_{kj} + H_j^l \bar{g}_{il}) = 2\bar{g}^{ij} H_i^k \bar{g}_{kj} = 2H_i^i.$$

We recall that the quantity H_i^i represents the opposite of the mean curvature of $\partial\Omega$ (by our choice of ν), and in particular it is independent of the choice of coordinates.

Using (34), (35), writing formally $(w_0 + \varepsilon w_1)^p$ as $w_0^p + p\varepsilon w_0^{p-1} w_1 + o(\varepsilon)$, and expanding $-\Delta_{g_\varepsilon} u_{1,\varepsilon} + u_{1,\varepsilon} = u_{1,\varepsilon}^p$ up to the first order in ε , we obtain the following equation for w_1

$$(36) \quad \begin{cases} -w_1'' + w_1 - p w_0^{p-1} w_1 = H_1^1 w_0' & \text{in } \mathbb{R}_+, \\ w_1'(0) = 0. \end{cases}$$

By Proposition 2.1 and the subsequent comment, equation (36) can be solved for any right-hand side in $L^2(\mathbb{R}_+^2)$, as in this case (see equation (5)). The resulting expression for w_1 will be independent of the choice of coordinates on $\partial\Omega_\varepsilon$.

The construction of a better approximate solution will be performed expanding formally the equation $-\Delta_{g_\varepsilon} u + u = u^p$ in powers of ε up to order k . A rigorous proof will be given in the next subsection.

3.2 Proof of Proposition 3.1

We note that the metric g_{AB} is a function of the metric $\bar{g}_{ij}$, the operator H_j^i , and of the variable εx_n . Hence, fixing an integer k and using a Taylor expansion, we can write

$$(37) \quad \frac{1}{\sqrt{\det g}} \partial_n \sqrt{\det g} = \sum_{i=1}^k \varepsilon^i x_n^{i-1} \tilde{G}_i(\varepsilon x') + \tilde{G}(\varepsilon x', x_n),$$

where $\tilde{G}_i : \partial\Omega \rightarrow \mathbb{R}$ are smooth functions, and $\tilde{G}$ satisfies

$$(38) \quad \left\| \tilde{G}(\cdot, x_n) \right\|'_{C^m(\partial\Omega)} \leq C_{k,m} x_n^k \varepsilon^{k+1}, \quad x_n \in J_\varepsilon,$$

where $C_{k,m}$ is a constant depending only on Ω , p , k , and m . Again (and in the following), when we write $\|\cdot\|'$ we keep the variable x_n fixed. We remark that, from the computations of the previous subsection, the function $\tilde{G}_1$ is the opposite of the mean curvature of $\partial\Omega$.

We need now a similar expansion for the operator Δ_g . In order to get expressions which are independent of the coordinates x' , we introduce some notation. Fixing the variable $x_n \in J_\varepsilon$, the metric $g(x', x_n) = g_\varepsilon(x', x_n)$ induces a metric $\hat{g}_{\varepsilon, x_n}$ on $\partial\Omega$ in the following way. Consider the homothety $i_\varepsilon : \partial\Omega \rightarrow \partial\Omega_\varepsilon$. We define $\hat{g}_{\varepsilon, x_n}$ to be

$$\hat{g}_{\varepsilon, x_n} = \varepsilon^2 i_\varepsilon^* g_\varepsilon(\cdot, x_n),$$

where i_ε^* denotes the pull-back operator. Note that, with the above scaling, the metric $\hat{g}_{\varepsilon, x_n}$ converges to $\hat{g}$ as ε goes to zero. More precisely, fixing an integer k , for any smooth function $f : \partial\Omega \rightarrow \mathbb{R}$ we have the expansion

$$(39) \quad -\Delta_{\hat{g}_{\varepsilon, x_n}} f = -\sum_{i=0}^k (\varepsilon x_n)^i L_i f + L_{\varepsilon, x_n} f,$$

where $\{L_i\}_i, L_{\varepsilon, x_n}$ are linear second-order differential operators (e.g. $L_0 = \Delta_{\hat{g}}$) satisfying

$$(40) \quad \|L_i f\|'_{C^m(\partial\Omega)} \leq C_m \|f\|'_{C^{m+2}(\partial\Omega)}; \quad \|L_{\varepsilon, x_n} f\|'_{C^m(\partial\Omega)} \leq C_m (\varepsilon x_n)^{k+1} \|f\|'_{C^{m+2}(\partial\Omega)}$$

for all smooth f , where C_m is a constant depending only on Ω , p , k , and m .

Consider now a function $u : S_\varepsilon \rightarrow \mathbb{R}$, which is of the form

$$(41) \quad u(x', x_n) = \tilde{u}(\varepsilon x', x_n), \quad x' \in \partial\Omega_\varepsilon, x_n \in J_\varepsilon,$$

see the notation of (33). Then, scaling in the variable x' , we have

$$-\Delta_{g_\varepsilon} u(x', x_n) = -\tilde{u}_{nn}(\varepsilon x', x_n) - \frac{1}{\sqrt{\det g}} \partial_n \left(\sqrt{\det g} \right) \tilde{u}_n(\varepsilon x', x_n) - \varepsilon^2 \Delta_{\hat{g}_{\varepsilon, x_n}} \tilde{u}(\varepsilon x', x_n).$$

Using the expansions (37), (39), the last equation becomes

$$(42) \quad \begin{aligned} -\Delta_{g_\varepsilon} u(x', x_n) &= -\tilde{u}_{nn}(\varepsilon x', x_n) - \sum_{i=1}^k \varepsilon^i x_n^{i-1} \tilde{G}_i \tilde{u}_n(\varepsilon x', x_n) - \sum_{i=0}^k \varepsilon^{2+i} x_n^i L_i \tilde{u}(\varepsilon x', x_n) \\ &\quad - \tilde{G}(\varepsilon x', x_n) \tilde{u}(\varepsilon x', x_n) - \varepsilon^2 L_{\varepsilon, x_n} \tilde{u}(\varepsilon x', x_n). \end{aligned}$$

This formula has the advantage of being independent of the choice of coordinates x' .

We also need to expand in power series of ε expressions of the form $(a_0 + \varepsilon a_1 + \dots + \varepsilon^k a_k)^p$, where $a_0 > 0$. We can write

$$(a_0 + h)^p = a_0^p \left(1 + \frac{h}{a_0}\right)^p = a_0^p \left(1 + p \frac{h}{a_0} + \dots + c_{k,p} \left(\frac{h}{a_0}\right)^k + O\left(\left|\frac{h}{a_0}\right|^{k+1}\right)\right),$$

where $h = \varepsilon a_1 + \dots + \varepsilon^k a_k$. Separating the powers of ε , we have

$$(a_0 + h)^p = a_0^p \left[\sum_{i=0}^{k^2} \varepsilon^i \sum_{\substack{j_1, \dots, j_k \\ \sum l_{j_i} = i}} C_{i,j_1, \dots, j_k} a_0^{-\sum j_i} a_1^{j_1} \dots a_k^{j_k} + O\left(\left|\frac{h}{a_0}\right|^{k+1}\right) \right].$$

Hence, collecting the powers of ε up to order k , and using elementary inequalities, we get

$$(43) \quad \left| (a_0 + h)^p - a_0^p \sum_{i=0}^k \varepsilon^i \sum_{\substack{j_1, \dots, j_k \\ \sum l_{j_i} = i}} C_{i,j_1, \dots, j_k} a_0^{-\sum j_i} a_1^{j_1} \dots a_k^{j_k} \right| \leq C_{k,p} a_0^p \left[\varepsilon^{k+1} \left(1 + \frac{|a_1| + \dots + |a_k|}{a_0}\right)^{k^2} + \left|\frac{h}{a_0}\right|^{k+1} \right],$$

provided $\frac{h}{a_0}$ is uniformly bounded. Here $C_{k,p}$ depends on k and p only. For smooth functions $\tilde{w}_1, \dots, \tilde{w}_k : \partial\Omega \times \mathbb{R}_+$, let

$$\hat{u}_{k,\varepsilon}(x', x_n) = w_0(x_n) + \varepsilon \tilde{w}_1(\varepsilon x', x_n) + \dots + \varepsilon^k \tilde{w}_k(\varepsilon x', x_n); \quad x' \in \partial\Omega_\varepsilon, x_n \in \mathbb{R}_+.$$

Expanding formally the equation $-\Delta_g u + u = u^p$ (imposing Neumann boundary conditions) in powers of ε , and using formulas (42), (43), we find that $\tilde{w}_i$ satisfies the following recurrence formula for i running from 1 to k

$$(44) \quad \begin{cases} -\tilde{w}_i'' + \tilde{w}_i - p w_0^{p-1} \tilde{w}_i = \sum_{j=1}^i x_n^{j-1} \tilde{G}_i(x') \tilde{w}'_{i-j}(\varepsilon x', x_n) - \sum_{j=0}^{i-2} (L_{i-j-2} \tilde{w}_j)(\varepsilon x', x_n) \\ \quad + \sum_{\substack{j_1, \dots, j_{i-1} \\ \sum l_{j_i} = i}} C_{i,j_1, \dots, j_i} w_0^{p-\sum j_i} \tilde{w}_1^{j_1} \dots \tilde{w}_{i-1}^{j_{i-1}}(\varepsilon x', x_n); \\ \tilde{w}'_k(0) = 0, \end{cases} \quad \text{in } \mathbb{R}_+$$

where the derivatives are taken with respect to the variable x_n , and the coefficients $C_{i,j_1, \dots, j_i}$ are defined in (43).

We use formula (44) to define recursively the functions $\tilde{w}_1, \dots, \tilde{w}_k$. Notice that in the right-hand side of the ODE in (44) appear only the functions $\tilde{w}_1, \dots, \tilde{w}_{i-1}$, and that the operator on the left-hand side is invertible provided the right-hand side is in $L^2(\mathbb{R}_+)$, see Proposition 2.1 and the subsequent comment.

Proceeding by induction, using the regularity of the functions $\tilde{G}_i$, equation (40), and using standard estimates on solutions of ODE's, we find that there exist polynomials $P_i(t)$ such that

$$(45) \quad \begin{cases} \|(\nabla')^{(m)} \tilde{w}_i(x', x_n)\|' \leq C_{i,m} P_i(x_n) e^{-x_n}; \\ \|(\nabla')^{(m)} \tilde{w}'_i(x', x_n)\|' \leq C_{i,m} P_i(x_n) e^{-x_n}; \\ \|(\nabla')^{(m)} \tilde{w}''_i(x', x_n)\|' \leq C_{i,m} P_i(x_n) e^{-x_n}; \end{cases} \quad x' \in \partial\Omega_\varepsilon, x_n \in \mathbb{R}_+, m = 0, 1, \dots,$$

where ∇' denotes the derivative with respect to x' , and $C_{i,m}$ is a constant depending only on Ω , p , i and m . In particular these decay estimates ensure that the right-hand side in (44) belongs to $L^2(\mathbb{R}_+)$, and (44) is solvable for all $i = 1, \dots, k$.

We will set $u_{k,\varepsilon} = \varphi_\varepsilon \times (w_0 + \varepsilon w_1 + \dots \varepsilon^k w_k)$ for a suitable cut-off function φ_ε . We first establish what the support of φ_ε should be. Define

$$\hat{u}_{k,\varepsilon}(x', x_n) = w_0(x_n) + \varepsilon \tilde{w}_1(\varepsilon x', x_n) + \dots \varepsilon^k \tilde{w}_k(\varepsilon x', x_n), \quad x' \in \partial\Omega_\varepsilon, x_n \in \mathbb{R}_+.$$

Let $d_1, \dots, d_k$ denote the degrees of the polynomials $P_1, \dots, P_k$, see formula (45). From now on we choose γ satisfying

$$(46) \quad \gamma < \frac{1}{2} \min_{i=1, \dots, k} \left\{ \frac{i}{d_i} \right\}.$$

Then from (45) we find

$$(47) \quad \begin{cases} \|(\nabla')^{(m)} \varepsilon^i \tilde{w}_i(x', x_n)\|' \leq C_{i,m} \varepsilon^i P_i(x_n) e^{-x_n} \leq C_{i,m} \varepsilon^{\frac{i}{2}} e^{-x_n}; \\ \|(\nabla')^{(m)} \varepsilon^i \tilde{w}'_i(x', x_n)\|' \leq C_{i,m} \varepsilon^{\frac{i}{2}} e^{-x_n}; \\ \|(\nabla')^{(m)} \varepsilon^i \tilde{w}''_i(x', x_n)\|' \leq C_{i,m} \varepsilon^{\frac{i}{2}} e^{-x_n}; \end{cases} \quad \text{for } x_n \in J_\varepsilon.$$

Using equation (43), we have

$$(48) \quad \begin{aligned} & \left| (\hat{u}_{k,\varepsilon})^p - w_0^p \sum_{i=0}^k \varepsilon^i \sum_{\substack{j_1, \dots, j_k \\ \sum l_{j_l} = i}} C_{i,j_1, \dots, j_k} w_0^{-\sum j_l} \tilde{w}_1^{j_1} \dots \tilde{w}_k^{j_k} \right|(\varepsilon x', x_n) \\ & \leq C_{k,p} w_0^p \left[\varepsilon^{k+1} \left(1 + \frac{|\tilde{w}_1| + \dots + |\tilde{w}_k|}{w_0} \right)^{k^2} + \left| \frac{\hat{u}_{k,\varepsilon} - w_0}{w_0} \right|^{k+1} \right](\varepsilon x', x_n), \end{aligned}$$

provided the quantity $\left| \frac{\hat{u}_{k,\varepsilon} - w_0}{w_0} \right|$ is uniformly bounded. From the decay estimates (45) we deduce

$$(49) \quad \left(1 + \frac{|\tilde{w}_1| + \dots + |\tilde{w}_k|}{w_0} \right)^{k^2}(\varepsilon x', x_n) \leq C_{k,p} \left(1 + \sum_{i=1}^k C_{i,0} P_{i,0}(x_n) \right)^{k^2} \leq \tilde{P}_k(x_n);$$

$$(50) \quad \left| \frac{\hat{u}_{k,\varepsilon} - w_0}{w_0} \right|^{k+1} \leq C_{k,p} (\varepsilon P_{1,0}(x_n) + \dots \varepsilon^k P_{k,0}(x_n))^{k+1} \leq \varepsilon^{k+1} \hat{P}_k(x_n),$$

for suitable polynomials $\tilde{P}_k$ and $\hat{P}_k$. Hence, setting

$$\tilde{f}(x', x_n) = (\hat{u}_{k,\varepsilon})^p - w_0^p \sum_{i=0}^k \varepsilon^i \sum_{\substack{j_1, \dots, j_k \\ \sum l_{j_l} = i}} C_{i,j_1, \dots, j_k} w_0^{-\sum j_l} \tilde{w}_1^{j_1} \dots \tilde{w}_k^{j_k}(\varepsilon x', x_n),$$

from (5), (48), (49) and (50) we find

$$(51) \quad |\tilde{f}(x', x_n)| \leq C_{k,p} e^{-px_n} \varepsilon^{k+1} \left(\tilde{P}_k(x_n) + \hat{P}_k(x_n) \right),$$

provided the quantity $\left| \frac{\hat{u}_{k,\varepsilon} - w_0}{w_0} \right|$ is uniformly bounded. From equations (5) and (47), this is the case for $x_n \in J_\varepsilon$, hence (51) applies and we obtain

$$|\tilde{f}(x', x_n)| \leq C_{k,p} e^{-px_n} \varepsilon^{k+1} \left(\tilde{P}_k(x_n) + \hat{P}_k(x_n) \right); \quad \hat{u}_{k,\varepsilon} > 0 \quad \text{for } x_n \in J_\varepsilon.$$

By the construction of the functions $\tilde{w}_i$ we have

$$(52) \quad \begin{aligned} \hat{f} &:= -\Delta \hat{u}_{k,\varepsilon} + \hat{u}_{k,\varepsilon} - \hat{u}_{k,\varepsilon}^p = -\varepsilon^{k+1} x_n^{k-1} L_{k-1} \hat{u}_{k,\varepsilon} - \varepsilon^{k+2} x_n^k L_k \hat{u}_{k,\varepsilon} \\ &- \tilde{G}(\varepsilon x', x_n) - \varepsilon^2 L_{\varepsilon, x_n} \hat{u}_{k,\varepsilon} + \tilde{f}. \end{aligned}$$

From estimates (38), (40), (45) and (52) we deduce

$$|\hat{f}(x', x_n)| \leq C \varepsilon^{k+1} (1 + x_n)^{\hat{d}_k} e^{-x_n}; \quad x' \in \partial\Omega_\varepsilon, x_n \in J_\varepsilon,$$

where $\hat{d}_k$ depends only on k , and C depends on Ω , p and k .

For γ as in (46), choose φ_ε satisfying (17) and set

$$u_{k,\varepsilon}(\varepsilon x', x_n) = \varphi_\varepsilon(x_n) \hat{u}_{k,\varepsilon}(\varepsilon x', x_n).$$

Then $u_{k,\varepsilon}$ has compact support in Ω_ε , and the estimates in (31) readily follow from (17) and (47). Moreover there holds

$$\begin{aligned} \left| -\Delta u_{k,\varepsilon} + u_{k,\varepsilon} - u_{k,\varepsilon}^p \right| &\leq \varphi_\varepsilon \left| -\Delta \hat{u}_{k,\varepsilon} + \hat{u}_{k,\varepsilon} - \hat{u}_{k,\varepsilon}^p \right| + 2|\nabla \hat{u}_{k,\varepsilon}| |\nabla \varphi_\varepsilon| + \hat{u}_{k,\varepsilon} |\Delta \varphi_\varepsilon| + |\hat{f}| \\ &\leq C \varepsilon^{k+1} (1 + x_n)^{\hat{d}_k} e^{-x_n}. \end{aligned}$$

Finally, for any smooth function $v : \Omega_\varepsilon \rightarrow \mathbb{R}$, using the Hölder inequality and the last formula we get

$$|I'_\varepsilon(u_{k,\varepsilon})[v]| \leq \int_{\Omega_\varepsilon} \left| -\Delta u_{k,\varepsilon} + u_{k,\varepsilon} - u_{k,\varepsilon}^p \right| |v| \leq C \varepsilon^{k+1-\frac{n-1}{2}} \|v\|,$$

where C is a constant depending on Ω , p and k . Then the conclusion follows by considering the closure of the smooth functions in $H^1(\Omega_\varepsilon)$.

Remark 3.2 *It was pointed out in [34], Remark 4.4, that the solutions concentrating at $\partial\Omega$ are thickest at the points where the mean curvature is maximal. This can also be shown using the present approach. We are going to prove that, when H_i^i is negative, the function w_1 defined in (36) is positive at infinity. Then the claim will follow from the linearity of (36).*

We can assume that $H_i^i = -1$, so we are lead to study the problem

$$(53) \quad \begin{cases} -v'' + v - p w_0^{p-1} v = -w'_0 & \text{in } \mathbb{R}_+, \\ v'(0) = 0. \end{cases}$$

It is standard to show that the function v satisfies the following decay at infinity

$$(54) \quad |v(x)| \leq C x^d e^{-x}; \quad \text{for } x \gg 1,$$

where d is some positive integer.

There holds

$$v(x) = \int_0^{+\infty} \mathcal{K}(x, s) f(s) ds; \quad x > 0,$$

where

$$f(s) = -w'_0(s) + p w_0^{p-1}(s) v(s) \quad \text{and} \quad \mathcal{K}(x, s) = \begin{cases} e^{-s} \cosh x & \text{for } x \leq s, \\ \cosh s e^{-x} & \text{for } x > s. \end{cases}$$

Using this representation, together with the decay estimates (5), (54), one can show that $v(x) > 0$ for x sufficiently large. This concludes the proof.

Remark 3.3 *We note that, in order to construct the function $u_{k,\varepsilon}$, we used the smoothness of Ω in a crucial way, see in particular (38), (40). In the subcritical case the choice of k , depending on n and p , is given by formula (87). Hence, for n large, we require Ω to be at least of class C^r , where $r \sim \frac{5}{2}n$ for $p > 2$, and $r \sim \left(\frac{1}{p-1} + \frac{3}{2}\right)n$ for $p \leq 2$.*

4 Eigenvalues of T_{Σ_ε}

In this section we study the eigenvalue problem (57) below, which represents a refined approximation for the study of the linearization of $(\bar{P}_\varepsilon)$ at $u_{k,\varepsilon}$, see Proposition 5.1. The integer k will be chosen in Section 5.

Let Γ_ε be as in Section 3, and let Σ_ε be defined as

$$\Sigma_\varepsilon = \Gamma_\varepsilon(S_\varepsilon) := \Gamma_\varepsilon(\partial\Omega_\varepsilon \times J_\varepsilon) = \{x \in \mathbb{R}^n : \text{dist}(x, \partial\Omega_\varepsilon) < \varepsilon^{-\gamma}\}.$$

We endow Σ_ε with the metric g_ε induced by the inclusion in $\mathbb{R}^n$, and we define the functional space

$$H_{\Sigma_\varepsilon} = \{u \in H^1(\Sigma_\varepsilon) : u(\Gamma_\varepsilon(x', \varepsilon^{-\gamma})) = 0 \text{ for all } x' \in \partial\Omega_\varepsilon\},$$

with its natural norm

$$\|u\|_{H_{\Sigma_\varepsilon}}^2 = \int_{\Sigma_\varepsilon} (|\nabla_{g_\varepsilon} u|^2 + u^2); \quad u \in H_{\Sigma_\varepsilon},$$

and the corresponding scalar product $(\cdot, \cdot)_{H_{\Sigma_\varepsilon}}$. Using the expression of the metric g_ε , see equation (32) and the subsequent formulas, one finds

$$(55) \quad (1 - C\varepsilon^{1-\gamma}) \|u\|_{H_{S_\varepsilon}} \leq \|u\|_{H_{\Sigma_\varepsilon}} \leq (1 + C\varepsilon^{1-\gamma}) \|u\|_{H_{S_\varepsilon}}; \quad \text{for all } u \in H_{S_\varepsilon},$$

where C is a constant depending only on Ω .

Let $u_{k,\varepsilon}$ be the function constructed in Section 3. Then we define $T_{\Sigma_\varepsilon} : H_{\Sigma_\varepsilon} \rightarrow H_{\Sigma_\varepsilon}$ in the following way

$$(56) \quad (T_{\Sigma_\varepsilon} u, v)_{H_{\Sigma_\varepsilon}} = \int_{\Sigma_\varepsilon} (\nabla_{g_\varepsilon} u \cdot \nabla_{g_\varepsilon} v + uv - pu_{k,\varepsilon}^{p-1} uv) dV_{g_\varepsilon}; \quad u, v \in H_{\Sigma_\varepsilon}.$$

The goal of this section is to study the eigenvalues σ of the problem

$$(57) \quad T_{\Sigma_\varepsilon} u = \sigma u, \quad u \in H_{\Sigma_\varepsilon}.$$

In the next Proposition we characterize the eigenfunctions of T_{Σ_ε} when the eigenvalue σ is close to zero. We recall the definition of the functions ϕ_l and $v_{l,\varepsilon}$ from Section 2.

Proposition 4.1 *Let $\gamma > 0$ be as in (46), and let $\Lambda \in (0, \tau/4)$. Suppose $\sigma \in [-\tau/4, \tau/4]$, and let u be an eigenfunction of T_{Σ_ε} with eigenvalue σ . Then there holds*

$$(58) \quad \left\| u - \sum_{\{l: \mu_{l,\varepsilon} \in [\sigma - \Lambda, \sigma + \Lambda]\}} \alpha_l \phi_l v_{l,\varepsilon} \right\|_{H_{\Sigma_\varepsilon}} \leq C \frac{\varepsilon^{1-\gamma}}{\Lambda} \|u\|_{H_{\Sigma_\varepsilon}},$$

for some coefficients $\{\alpha_l\}$ and for some constant C depending only on Ω and p .

PROOF. From (56), the eigenvalue equation (57) can be written as

$$(1 - \sigma) \int_{\Sigma_\varepsilon} (\nabla_{g_\varepsilon} u \cdot \nabla_{g_\varepsilon} v + uv) dV_{g_\varepsilon} - p \int_{\Sigma_\varepsilon} u_{k,\varepsilon}^{p-1} uv dV_{g_\varepsilon} = 0 \quad \text{for all } v \in H_{\Sigma_\varepsilon}.$$

Using (32), the construction of $u_{k,\varepsilon}$ and (55), the above equation becomes

$$(59) \quad \left| (1 - \sigma) \int_{S_\varepsilon} (\nabla_{g_0} u \cdot \nabla_{g_0} v + uv) dV_{g_0} - p \int_{S_\varepsilon} u_0^{p-1} uv dV_{g_0} \right| \leq C \varepsilon^{1-\gamma} \|u\|_{H_{S_\varepsilon}} \|v\|_{H_{S_\varepsilon}} \quad \text{for all } v \in H_{S_\varepsilon},$$

where C is a constant depending only on Ω and p . The metric g_0 was defined in (22). We decompose u and v uniquely as

$$u(x', x_n) = \sum_{l=0}^{\infty} \phi_l(\varepsilon x') (\alpha_l v_{l,\varepsilon}(x_n) + \hat{u}_l(x_n)); \quad v(x', x_n) = \sum_{l=0}^{\infty} \phi_l(\varepsilon x') (\beta_l v_{l,\varepsilon}(x_n) + \hat{v}_l(x_n)),$$

where $\{\alpha_l\}_l, \{\beta_l\}_l$ are real numbers, and where

$$(\hat{u}_l, v_{l,\varepsilon})_{l,\varepsilon} = 0; \quad (\hat{v}_l, v_{l,\varepsilon})_{l,\varepsilon} = 0 \quad \text{for all } l.$$

From (24) we have

$$(60) \quad \|u\|_{H_{S_\varepsilon}}^2 = \frac{1}{\varepsilon^{n-1}} \sum_l (\alpha_l^2 + \|\hat{u}_l\|_{l,\varepsilon}^2); \quad \|v\|_{H_{S_\varepsilon}}^2 = \frac{1}{\varepsilon^{n-1}} \sum_l (\beta_l^2 + \|\hat{v}_l\|_{l,\varepsilon}^2),$$

and similarly

$$(61) \quad \int_{S_\varepsilon} (\nabla_{g_0} u \cdot \nabla_{g_0} v + uv) dV_{g_0} - p \int_{S_\varepsilon} w_0^{p-1} uv dV_{g_0} = \frac{1}{\varepsilon^{n-1}} \sum_l (\alpha_l \beta_l \mu_{l,\varepsilon} + (T_{l,\varepsilon} \hat{u}_l, \hat{v}_l)_{l,\varepsilon}).$$

Using (60) and (61), (59) reads

$$(62) \quad \left| \sum_l ((\mu_{l,\varepsilon} - \sigma) \alpha_l \beta_l + ((T_{l,\varepsilon} - \sigma) \hat{u}_l, \hat{v}_l)) \right| \leq C \varepsilon^{1-\gamma} \left(\sum_l \alpha_l^2 + \|\hat{u}_l\|_{l,\varepsilon}^2 \right)^{\frac{1}{2}} \left(\sum_l \beta_l^2 + \|\hat{v}_l\|_{l,\varepsilon}^2 \right)^{\frac{1}{2}} \quad \text{for all } v \in H_{S_\varepsilon}.$$

We can write $u = u_1 + u_2$, where

$$u_1 = \sum_{l=0}^{\infty} \alpha_l v_{l,\varepsilon} \phi_l; \quad u_2 = \sum_{l=0}^{\infty} \hat{u}_l \phi_l.$$

Recalling that $((T_{l,\varepsilon} - \sigma) \hat{u}_l, \hat{u}_l) \geq \frac{\tau}{2} \|\hat{u}_l\|_{l,\varepsilon}^2$ (by (29) and since $\sigma \in [-\tau/4, \tau/4]$), and taking $v = u_2$ in (62), we get

$$(63) \quad \frac{\tau}{2} \|u_2\|_{H_{S_\varepsilon}}^2 \leq C \varepsilon^{1-\gamma} \|u\|_{H_{S_\varepsilon}}^2.$$

We also write $u_1 = u_- + u_0 + u_+$, where

$$u_- = \sum_{\{l: \mu_{l,\varepsilon} < \sigma - \Lambda\}} \alpha_l \phi_l v_{l,\varepsilon}; \quad u_0 = \sum_{\{l: \mu_{l,\varepsilon} \in [\sigma - \Lambda, \sigma + \Lambda]\}} \alpha_l \phi_l v_{l,\varepsilon}; \quad u_+ = \sum_{\{l: \mu_{l,\varepsilon} > \sigma + \Lambda\}} \alpha_l \phi_l v_{l,\varepsilon}.$$

Taking $v = u_-, u_+$ in (62), we finally obtain

$$(64) \quad \|u_-\|_{H_{S_\varepsilon}}^2 \leq C \frac{\varepsilon^{1-\gamma}}{\Lambda} \|u\|_{H_{S_\varepsilon}}^2; \quad \|u_+\|_{H_{S_\varepsilon}}^2 \leq C \frac{\varepsilon^{1-\gamma}}{\Lambda} \|u\|_{H_{S_\varepsilon}}^2.$$

Then the conclusion follows from (55), (63) and (64). ■

Remark 4.2 In [34] there is an analogue of Proposition 4.1, based on cancellation of oscillatory integrals. Although more involved, it showed a strong localization in the Fourier modes of the eigenfunctions for $n = 2$. We believe the same phenomenon should occur in higher dimensions. To prove such an estimate, it is useful to control the integrals of products of three eigenfunctions of $\Delta_{\hat{g}}$. When $n = 2$, namely when $\partial\Omega$ is 1-dimensional, this follows from simple trigonometric formulas.

Let $\bar{u}_{k,\varepsilon} : \Omega \rightarrow \mathbb{R}$ the function be the function defined by

$$\bar{u}_{k,\varepsilon}(x) = u_{k,\varepsilon}(\varepsilon x), \quad x \in \Omega.$$

We recall the following result from [34].

Proposition 4.3 *The eigenvalues of the operator T_{Σ_ε} are differentiable with respect to ε , the eigenvalues being considered a possibly multivalued function of ε . If $\sigma(\varepsilon)$ is such an eigenvalue, then*

$$(65) \quad \frac{\partial \sigma}{\partial \varepsilon} = \{\text{eigenvalues of } Q_\sigma\},$$

where $Q_\sigma : H_\sigma \times H_\sigma \rightarrow \mathbb{R}$ is the quadratic form given by

$$(66) \quad Q_\sigma(u, v) = (1 - \sigma) \frac{2}{\varepsilon} \int_{\Sigma_\varepsilon} \nabla u \cdot \nabla v - p(p-1) \int_{\Sigma_\varepsilon} uv u_{k,\varepsilon}^{p-2} \frac{\partial \bar{u}_{k,\varepsilon}}{\partial \varepsilon}(\varepsilon \cdot).$$

Here $H_\sigma \subseteq H_{\Sigma_\varepsilon}$ denotes the eigenspace of T_{Σ_ε} corresponding to σ and the function $u_{k,\varepsilon}$ is defined in Section 3.

We apply Proposition 4.3 to the eigenvalues σ which are close to 0.

Proposition 4.4 *Let σ be an eigenvalue of T_{Σ_ε} belonging to the interval $[-\tau/4, \tau/4]$. Then there holds*

$$\left| \frac{\partial \sigma}{\partial \varepsilon} - \frac{1}{\varepsilon} \tilde{F}(\sigma) \right| \leq C \varepsilon^{-\frac{1+\gamma}{2}},$$

where γ is as in (46), $\tilde{F}(\cdot)$ is given in (14), and C is a constant depending only on Ω and p .

PROOF. Recalling that $\frac{\partial \sigma}{\partial \varepsilon}$ can be a multivalued function, and using Proposition 4.3, it is sufficient to show that

$$(67) \quad \sup_{u \in H_\sigma} \frac{Q_\sigma(u, u)}{\|u\|^2} \leq \frac{2}{\varepsilon} (1 - \sigma) \tilde{F}(\sigma) + C \varepsilon^{-\frac{1+\gamma}{2}}; \quad \inf_{u \in H_\sigma} \frac{Q_\sigma(u, u)}{\|u\|^2} \geq \frac{2}{\varepsilon} (1 - \sigma) \tilde{F}(\sigma) - C \varepsilon^{-\frac{1+\gamma}{2}}.$$

From the expression of g_ε (formula (32)) and the properties of $u_{k,\varepsilon}$ (Proposition 3.1), we obtain

$$(68) \quad \left| \frac{2}{\varepsilon} (1 - \sigma) \int_{\Sigma_\varepsilon} |\nabla_{g_\varepsilon} u|^2 - \frac{2}{\varepsilon} (1 - \sigma) \int_{S_\varepsilon} |\nabla_{g_0} u|^2 dV_{g_0} \right| \leq C \varepsilon^{-\gamma} \|u\|_{\Sigma_\varepsilon}^2,$$

where C is a constant depending only on Ω and p .

Let u be an eigenfunction of T_{Σ_ε} with $\sigma \in [-\tau/4, \tau/4]$. Choosing $\Lambda \in [-\tau/4, \tau/4]$, equation (58) holds true and we can also write (by (24))

$$(69) \quad \left| \|u\|_{H_{\Sigma_\varepsilon}}^2 - \frac{1}{\varepsilon^{n-1}} \sum_{\{l: \mu_{l,\varepsilon} \in [\sigma - \Lambda, \sigma + \Lambda]\}} \alpha_l^2 \right| \leq C \frac{\varepsilon^{1-\gamma}}{\Lambda} \|u\|_{H_{\Sigma_\varepsilon}}^2, \quad u \in H_\sigma.$$

We can assume that the cut-off function φ_ε in (17) is of the form

$$\varphi_\varepsilon(x) = \varphi_0(\varepsilon^{1-\gamma} x),$$

where φ_0 is a smooth cut-off function in $[0, 1]$. Consider the neighborhood $\partial_\delta \Omega$

$$\partial_\delta \Omega = \{x \in \bar{\Omega} : \text{dist}(x, \partial \Omega) < \delta\},$$

and its parametrization $\tilde{\Gamma}_\delta : \partial \Omega \times [0, \delta]$ given by

$$\tilde{\Gamma}_\delta(y', y) = y' + y\nu; \quad y' \in \partial \Omega, y \in [0, \delta].$$

Using these coordinates we have

$$\bar{u}_{k,\varepsilon}(y', y) = \varphi_0(\varepsilon^{\gamma-1}y) [w_0(y/\varepsilon) + \varepsilon \tilde{w}_1(y', y/\varepsilon) + \cdots + \varepsilon^k \tilde{w}_k(y', y/\varepsilon)],$$

where the functions $\tilde{w}_1, \dots, \tilde{w}_k$ are defined in Section 3. Differentiating with respect to ε and scaling back to Ω_ε , one finds

$$\begin{aligned} \frac{\partial \bar{u}_{k,\varepsilon}}{\partial \varepsilon}(x', x_n) &= (\gamma - 1)\varepsilon^{\gamma-2} \frac{\partial \varphi_\varepsilon}{\partial x_n} [w_0(x_n) + \varepsilon \tilde{w}_1(\varepsilon x', x_n) + \cdots + \varepsilon^k \tilde{w}_k(\varepsilon x', x_n)] \\ &\quad - \varphi_\varepsilon \frac{x_n}{\varepsilon} \left[\frac{\partial w_0}{\partial x_n}(x_n) + \varepsilon \frac{\partial \tilde{w}_1}{\partial x_n}(\varepsilon x', x_n) + \cdots + \varepsilon^k \frac{\partial \tilde{w}_k}{\partial x_n}(\varepsilon x', x_n) \right] \\ &\quad + \varphi_\varepsilon [\tilde{w}_1(\varepsilon x', x_n) + \cdots + k\varepsilon^{k-1} \tilde{w}_k(\varepsilon x', x_n)]; \quad x' \in \partial\Omega_\varepsilon, x_n \in J_\varepsilon. \end{aligned}$$

From the estimates in Section 3 it follows that

$$\int_{\Sigma_\varepsilon} uv u_{k,\varepsilon}^{p-2} \frac{\partial \bar{u}_{k,\varepsilon}}{\partial \varepsilon}(\varepsilon \cdot) = -\frac{1}{\varepsilon} \int_{S_\varepsilon} x_n uv w_0^{p-2} w'_0 dV_{g_0} + O(\varepsilon^{-\gamma}) \|u\|_{H_{\Sigma_\varepsilon}} \|v\|_{H_{\Sigma_\varepsilon}}; \quad u, v \in H_{\Sigma_\varepsilon}.$$

The last estimate, with equations (68), (69) and the strict monotonicity of $\alpha \mapsto \mu_\alpha$ yields

$$\begin{aligned} Q_\sigma(u, u) &= \frac{2}{\varepsilon} \frac{1}{\varepsilon^{n-1}} \sum_{\{l: \mu_{l,\varepsilon} \in [\sigma - \Lambda, \sigma + \Lambda]\}} (1 - \mu_{l,\varepsilon}) \alpha_l^2 \int_{J_\varepsilon} ((v'_{l,\varepsilon})^2 + \varepsilon^2 \lambda_l v_{l,\varepsilon}^2) \\ &\quad + \frac{p(p-1)}{\varepsilon} \frac{1}{\varepsilon^{n-1}} \sum_{\{l: \mu_{l,\varepsilon} \in [\sigma - \Lambda, \sigma + \Lambda]\}} \alpha_l^2 \int_{J_\varepsilon} (x_n w_0^{p-2} w'_0 v_{l,\varepsilon}^2) + O\left(\frac{\varepsilon^{-\gamma}}{\Lambda}\right) \|u\|_{\Sigma_\varepsilon}^2, \end{aligned}$$

for all $u \in H_\sigma$. Using a Pohozaev-type identity, see also [34] pag. 1554, one finds

$$2(1 - \mu_\alpha) \int_{\mathbb{R}_+} ((v'_\alpha)^2 + \alpha v_\alpha^2) + p(p-1) \int_{\mathbb{R}_+} v_\alpha^2 x_n w_0^{p-2} w'_0 = \tilde{F}(\alpha).$$

From the last two equations, Proposition 2.3, and the smoothness of $\tilde{F}(\cdot)$ we have

$$(70) \quad Q_\sigma(u, u) \leq \frac{2}{\varepsilon} (1 - \sigma) \frac{1}{\varepsilon^{n-1}} \sum_{\{l: \mu_{l,\varepsilon} \in [\sigma - \Lambda, \sigma + \Lambda]\}} \alpha_l^2 \left(\tilde{F}(\sigma) + C(\Lambda + \varepsilon^{-\frac{\gamma}{2}} e^{-\delta \varepsilon^{-\gamma}}) \right) + C \frac{\varepsilon^{-\gamma}}{\Lambda} \|u\|_{\Sigma_\varepsilon}^2.$$

Taking $\Lambda = \varepsilon^{\frac{1-\gamma}{2}}$, from (69) and (70) we achieve

$$\sup_{u \in H_\sigma} \frac{Q_\sigma(u, u)}{\|u\|^2} \leq \frac{2}{\varepsilon} (1 - \sigma) \tilde{F}(\sigma) + C \varepsilon^{-\frac{1+\gamma}{2}},$$

namely the first inequality in (67). The second one can be obtained in the same way. ■

Now we are in position to prove the following proposition, which characterizes the spectrum of T_{Σ_ε} .

Proposition 4.5 *Let $u_{k,\varepsilon}$ and T_{Σ_ε} be as above. Then there exists a positive constant C , depending on p , Ω and k with the following property. For a suitable sequence $\varepsilon_j \rightarrow 0$, the operator $T_{\Sigma_{\varepsilon_j}} : H_{\Sigma_{\varepsilon_j}} \rightarrow H_{\Sigma_{\varepsilon_j}}$ is invertible and the inverse operator satisfies*

$$\left\| T_{\Sigma_{\varepsilon_j}}^{-1} \right\| \leq \frac{C}{\varepsilon_j^{n-1}}, \quad \text{for all } j \in \mathbb{N}.$$

PROOF. Let $\{\sigma_j\}_j$ (resp. $\{s_j\}_j$) denote the eigenvalues of T_{Σ_ε} (resp. T_{S_ε}), counted in increasing order with their multiplicity. Let also

$$N(\Sigma_\varepsilon) = \#\{j : \sigma_j \leq 0\}; \quad (\text{resp. } N(S_\varepsilon) = \#\{j : s_j \leq 0\}).$$

denote the number of non-positive eigenvalues of T_{Σ_ε} (resp. T_{S_ε}). Recall that, from Proposition 2.5, the eigenvalues of T_{S_ε} coincide with the numbers $\{\mu_{j,\varepsilon}\}_j := \mu_{\varepsilon^2\lambda_j,\varepsilon}$, where $\{\lambda_j\}_j$ are the eigenvalues of the Laplace-Beltrami operator $-\Delta_{\hat{g}}$, see also equations (23), (26). We also recall the Weyl's asymptotic formula, see e.g. [13] page 169

$$(71) \quad \lambda_j \sim \frac{(2\pi)^2}{\omega_{\frac{n-1}{2}}} \left(\frac{j}{\text{Vol}(\partial\Omega)} \right)^{\frac{2}{n-1}} = C_{n,\Omega} j^{\frac{2}{n-1}}, \quad \text{as } j \rightarrow +\infty.$$

From the strict monotonicity of $\alpha \mapsto \mu_\alpha := \mu(\alpha)$ and from the last formula we deduce

$$(72) \quad \#\{j : \varepsilon^2\lambda_j \leq \mu^{-1}(0)\} \sim \tilde{C}_{n,\Omega,p}\varepsilon^{1-n}.$$

From Proposition 2.3, we have

$$(73) \quad |\mu_{\varepsilon^2\lambda_j,\varepsilon} - \mu_{\varepsilon^2\lambda_j}| \leq C e^{-\delta\varepsilon^{-\gamma}}; \quad \text{provided } \mu_{\varepsilon^2\lambda_j} \in \left[-\frac{\tau}{4}, \frac{\tau}{4}\right],$$

which implies

$$\#\{j : \varepsilon^2\lambda_j \leq \mu^{-1}(0) - C e^{-\delta\varepsilon^{-\gamma}}\} \leq N(S_\varepsilon) \leq \#\{j : \varepsilon^2\lambda_j \leq \mu^{-1}(0) + C e^{-\delta\varepsilon^{-\gamma}}\}$$

and hence

$$(74) \quad N(S_\varepsilon) \sim \tilde{C}_{n,\Omega,p}\varepsilon^{1-n}.$$

From the Courant-Fischer method, we get

$$\sigma_j = \inf \left\{ \sup_{u \in M} \frac{(T_{\Sigma_\varepsilon} u, u)_{H_{\Sigma_\varepsilon}}}{\|u\|_{H_{\Sigma_\varepsilon}}^2} : M \text{ subspace of } H_{\Sigma_\varepsilon}, \dim M = j \right\},$$

and a similar characterization for s_j . There holds

$$|(T_{\Sigma_\varepsilon} u, u)_{H_{\Sigma_\varepsilon}} - (T_{S_\varepsilon} u, u)_{H_{S_\varepsilon}}| \leq C \varepsilon^{1-\gamma} \|u\|_{H_{S_\varepsilon}}^2; \quad \left| \|u\|_{H_{\Sigma_\varepsilon}}^2 - \|u\|_{H_{S_\varepsilon}}^2 \right| \leq C \varepsilon^{1-\gamma} \|u\|_{H_{S_\varepsilon}}^2,$$

for all $u \in H_{\Sigma_\varepsilon}$ and for some constant C depending only on Ω and p . Note that, by (55), in the last inequality we can also substitute $\|\cdot\|_{H_{S_\varepsilon}}$ with $\|\cdot\|_{H_{\Sigma_\varepsilon}}$. The last two formulas yield

$$(75) \quad |s_j - \sigma_j| \leq C \varepsilon^{1-\gamma}, \quad \text{for all } j,$$

where C depends only on Ω and p . From (73), (75) there exists a positive constant C , depending only on Ω and p , such that

$$\#\{j : \varepsilon^2\lambda_j \leq \mu^{-1}(0) - C \varepsilon^{1-\gamma}\} \leq N(\Sigma_\varepsilon) \leq \#\{j : \varepsilon^2\lambda_j \leq \mu^{-1}(0) + C \varepsilon^{1-\gamma}\},$$

which implies

$$(76) \quad N(\Sigma_\varepsilon) \sim \#\left\{j : \lambda_j \leq \frac{\mu^{-1}(0)}{\varepsilon^2}\right\} \sim \left(\frac{\mu^{-1}(0)}{\varepsilon^2 C_{n,\Omega}}\right)^{\frac{n-1}{2}} = \tilde{C}_{n,\Omega,p}\varepsilon^{1-n}.$$

For $l \in \mathbb{N}$, let $\varepsilon_l = 2^{-l}$. Then from (76) we have

$$(77) \quad N(\Sigma_{\varepsilon_{l+1}}) - N(\Sigma_{\varepsilon_l}) \sim \tilde{C}_{n,\Omega,p} \left(2^{(l+1)(n-1)} - 2^{l(n-1)} \right) = \tilde{C}_{n,\Omega,p} (2^{n-1} - 1) \varepsilon_l^{1-n}.$$

Note that, by Proposition 4.4, the eigenvalues of T_{Σ_ε} close to 0 decrease when ε decreases to zero. In other words, by the last equation, the number of eigenvalues which cross 0, when ε decreases from ε_l to ε_{l+1} , is of order ε_l^{1-n} . Now we define

$$A_l = \{\varepsilon \in (\varepsilon_{l+1}, \varepsilon_l) : \ker T_{\Sigma_\varepsilon} \neq \emptyset\}; \quad B_l = (\varepsilon_{l+1}, \varepsilon_l) \setminus A_l.$$

By Proposition 4.4 and (77), it follows that $\text{card}(A_l) < C\varepsilon_l^{1-n}$, and hence there exists an interval (a_l, b_l) such that

$$(78) \quad (a_l, b_l) \subseteq B_l; \quad |b_l - a_l| \geq C^{-1} \frac{\text{meas}(B_l)}{\text{card}(A_l)} \geq C^{-1} \varepsilon_l^n.$$

From Proposition 4.4 we deduce that

$$T_{\Sigma_{\frac{a_l+b_l}{2}}} \quad \text{is invertible and} \quad \left\| T_{\Sigma_{\frac{a_l+b_l}{2}}}^{-1} \right\| \leq \frac{C}{\varepsilon_l^{n-1}}.$$

Now it is sufficient to set $\varepsilon_j = \frac{a_j+b_j}{2}$. This concludes the proof. ■

5 Proof of Theorem 1.1

In this section we prove Theorem 1.1. We divide the proof in the two cases $p \leq \frac{n+2}{n-2}$ and $p > \frac{n+2}{n-2}$.

5.1 The case $p \leq \frac{n+2}{n-2}$

The following result can be proved as is [34], Proposition 5.6.

Proposition 5.1 *Let $k \in \mathbb{N}$ be fixed, and let $u_{k,\varepsilon}$ be as in Section 3. Let $\{\sigma_j\}_j$ denote the eigenvalues of T_{Σ_ε} , counted in increasing order and with their multiplicity, and let $\{\hat{\sigma}_j\}_j$ denote the eigenvalues of $I''_\varepsilon(u_{k,\varepsilon})$. Then there exists a positive constant C , depending only on Ω and p , such that*

$$|\sigma_j - \hat{\sigma}_j| \leq C e^{-\delta \varepsilon^{-\frac{\gamma}{2}}}; \quad \text{whenever } \sigma_j \in \left[-\frac{\tau}{4}, \frac{\tau}{4}\right].$$

The proof consists in showing that the eigenfunctions of $I''_\varepsilon(u_{k,\varepsilon})$ decay exponentially away from $\partial\Omega_\varepsilon$, and multiplying by suitable cut-off functions. The eigenvalues σ_j and $\hat{\sigma}_j$ are then compared using the Courant-Fisher method, as in the proof of Proposition 2.3.

Let ε_j be as in Proposition 4.5. Then from Proposition 5.1 we deduce

$$(79) \quad I''_{\varepsilon_j}(u_{k,\varepsilon_j}) \quad \text{is invertible and} \quad \|I''_{\varepsilon_j}(u_{k,\varepsilon_j})^{-1}\| \leq \frac{C}{\varepsilon_j^{n-1}}.$$

For brevity, in the rest of the proof, we simply write ε instead of ε_j . For $p \leq \frac{n+2}{n-2}$ we just apply the contraction mapping theorem, looking for a solution u_ε of the form

$$u_\varepsilon = u_{k,\varepsilon} + w, \quad w \in H^1(\Omega_\varepsilon).$$

If $I''_\varepsilon(u_{k,\varepsilon})$ is invertible (which is true along the sequence ε_j), we can write

$$(80) \quad I''_\varepsilon(u_{k,\varepsilon} + w) = 0 \quad \Leftrightarrow \quad w = -(I''_\varepsilon(u_{k,\varepsilon}))^{-1} [I'_\varepsilon(u_{k,\varepsilon}) + G(w)],$$

where

$$G(w) = I'_\varepsilon(u_{k,\varepsilon} + w) - I'_\varepsilon(u_{k,\varepsilon}) - I''_\varepsilon(u_{k,\varepsilon})[w].$$

Note that

$$G(w)[v] = - \int_{\Omega_\varepsilon} \left[(u_{k,\varepsilon} + w)^p - u_{k,\varepsilon}^p - p u_{k,\varepsilon}^{p-1} w \right] v; \quad v \in H^1(\Omega_\varepsilon).$$

Since the function $u_{k,\varepsilon}$ is uniformly bounded, we have the following elementary inequalities

$$(81) \quad \left| (u_{k,\varepsilon} + w)^p - u_{k,\varepsilon}^p - p u_{k,\varepsilon}^{p-1} w \right| \leq \begin{cases} C|w|^p & \text{for } p \leq 2, \\ C(|w|^2 + |w|^p) & \text{for } p > 2; \end{cases} \quad w \in \mathbb{R}$$

$$(82) \quad \begin{aligned} & \left| (u_{k,\varepsilon} + w_1)^p - (u_{k,\varepsilon} + w_2)^p - p u_{k,\varepsilon}^{p-1} (w_1 - w_2) \right| \\ & \leq \begin{cases} C (|w_1|^{p-1} + |w_2|^{p-1}) |w_1 - w_2| & \text{for } p \leq 2, \\ C (|w_1|^{p-1} + |w_2|^{p-1} + |w_1| + |w_2|) & \text{for } p > 2; \end{cases} \quad w_1, w_2 \in \mathbb{R}, \end{aligned}$$

where the constant C depends only on p . These inequalities and the Sobolev embeddings imply

$$(83) \quad \|G(w)\| \leq \begin{cases} C \|w\|^p & \text{for } p \leq 2; \\ C \|w\|^2 & \text{for } p > 2; \end{cases} \quad \|w\| \leq 1;$$

$$(84) \quad \|G(w_1) - G(w_2)\| \leq \begin{cases} C (\|w_1\|^{p-1} + \|w_2\|^{p-1}) \|w_1 - w_2\| & p \leq 2, \\ C (\|w_1\| + \|w_2\|) \|w_1 - w_2\| & p > 2; \end{cases} \quad \|w_1\|, \|w_2\| \leq 1.$$

Let us define the operator $F_\varepsilon : H^1(\Omega_\varepsilon) \rightarrow H^1(\Omega_\varepsilon)$

$$F_\varepsilon(w) = - (I_\varepsilon''(u_{k,\varepsilon}))^{-1} [I_\varepsilon'(u_{k,\varepsilon}) + G(w)], \quad w \in H^1(\Omega_\varepsilon).$$

Let ε_j be given by Proposition 4.5. We are going to prove that F_ε is a contraction in a suitable closed set of $H^1(\Omega_\varepsilon)$. From Proposition 3.1, (79) and (83), (84) we get

$$(85) \quad \|F_{\varepsilon_j}(w)\| \leq \begin{cases} C \varepsilon_j^{-(n-1)} \left(\varepsilon_j^{k+1-\frac{n-1}{2}} + \|w\|^p \right) & \text{for } p \leq 2, \\ C \varepsilon_j^{-(n-1)} \left(\varepsilon_j^{k+1-\frac{n-1}{2}} + \|w\|^2 \right) & \text{for } p > 2; \end{cases} \quad \|w\| \leq 1;$$

$$(86) \quad \|F_{\varepsilon_j}(w_1) - F_{\varepsilon_j}(w_2)\| \leq \begin{cases} C \varepsilon_j^{-(n-1)} (\|w_1\|^{p-1} + \|w_2\|^{p-1}) \|w_1 - w_2\| & p \leq 2, \\ C \varepsilon_j^{-(n-1)} (\|w_1\| + \|w_2\|) \|w_1 - w_2\| & p > 2; \end{cases} \quad \|w_1\|, \|w_2\| \leq 1.$$

Now we choose integers d and k such that

$$(87) \quad d > \begin{cases} \frac{n-1}{p-1} & \text{for } p \leq 2, \\ n-1 & \text{for } p > 2; \end{cases} \quad k+1 > d + \frac{3}{2}(n-1),$$

and we set

$$\mathcal{B}_j = \{w \in H^1(\Omega_\varepsilon) : \|w\| \leq \varepsilon_j^d\}.$$

From (85), (86) it follows that F_{ε_j} is a contraction in $\mathcal{B}_j$ for ε_j small. Since u_ε is close in norm to a positive function $(u_{k,\varepsilon})$, the positivity of u_ε follows from standard arguments. The points **i**) and **ii**) in the statement follow from the construction of u_ε . This concludes the proof in the case $p \leq \frac{n+2}{n-2}$.

Remark 5.2 By the above construction and some standard estimates, there holds

$$I_{\varepsilon_j}''(u_{\varepsilon_j}) = I_{\varepsilon_j}''(u_{k,\varepsilon_j}) + O\left(\varepsilon_j^{\min\{1,p-1\}}\right) = I_{\varepsilon_j}''(u_{k,\varepsilon_j}) + o(\varepsilon_j^n).$$

It follows from Propositions 4.5, 5.1 that the Morse index of u_{ε_j} coincides with $N(\Sigma_{\varepsilon_j}) \sim \varepsilon_j^{1-n}$.

Remark 5.3 Let ε_l be as above, and let C be a given constant. We define the set

$$D_l = \{\varepsilon \in (\varepsilon_{l+1}, \varepsilon_l) : T_{\Sigma_\varepsilon} \text{ is invertible and } \|T_{\Sigma_\varepsilon}^{-1}\| \leq C\varepsilon^{-n}\}.$$

Note that we are requiring an upper bound on $\|T_{\Sigma_\varepsilon}^{-1}\|$ which is much bigger than the one obtained before. However, choosing the integers k and d appropriately, one still can apply the Contraction Mapping Theorem for $\varepsilon \in D_l$.

From the proof of Proposition 4.5 it follows that

$$\frac{\text{meas}(D_l)}{\varepsilon_l - \varepsilon_{l+1}} \rightarrow 0 \quad \text{as } l \rightarrow +\infty.$$

Hence we obtain existence of solutions of (P_ε) when ε belongs to a sequence of disjoint intervals $[c_j, d_j]$ for which

$$\lim_{\varepsilon \rightarrow 0^+} \frac{\text{meas}((\cup_j [c_j, d_j]) \cap [0, \varepsilon])}{\varepsilon} = 1.$$

This means that the density of the set of the parameters ε for which the method gives a solution is converging to 1.

5.2 The supercritical case

In this section we prove Theorem 1.1 in the supercritical case $p > \frac{n+2}{n-2}$. The proof relies on a combination of the previous norm estimates and Sobolev estimates from the theory of elliptic PDEs. For $\varepsilon = \varepsilon_j$, the above operator F_ε is well-defined on $H^1(\Omega_\varepsilon) \cap L^\infty(\Omega_\varepsilon)$. Letting $\tilde{w} = F(w)$, we have the following equation

$$(88) \quad \begin{cases} -\Delta \tilde{w} = pu_{k,\varepsilon}^{p-1} \tilde{w} - \tilde{w} + f_{k,\varepsilon} + g(w) & \text{in } \Omega_\varepsilon, \\ \frac{\partial \tilde{w}}{\partial \nu} = 0 & \text{on } \partial\Omega_\varepsilon, \end{cases}$$

where

$$f_{k,\varepsilon} = \Delta u_{k,\varepsilon} - u_{k,\varepsilon} + u_{k,\varepsilon}^p; \quad g(w) = (u_{k,\varepsilon} + w)^p - u_{k,\varepsilon}^p - pu_{k,\varepsilon}^{p-1}w,$$

together with the norm estimates (85)-(86).

We note that the function $\tilde{w}$ is of class $W^{2,2}$ in Ω_ε . This can be proved as follows. Using the Neumann boundary condition, one can extend $\tilde{w}$ as a function of class H^1 in a neighborhood of Ω_ε . Also, $u_{k,\varepsilon}$, $f_{k,\varepsilon}$ and $g(w)$ can be extended as L^2 functions in such a neighborhood. Then interior regularity estimates, see e.g. [23], Theorem 8.10, yield $\tilde{w} \in W^{2,2}(\Omega_\varepsilon)$.

Now, given $q > 1$, one can invoke Theorem 15.2 in [1] to get

$$(89) \quad \|\tilde{w}\|_{W^{2,q}(\Omega_\varepsilon)} \leq \frac{C}{\varepsilon^{\tilde{D}(n,q)}} (\|\tilde{w}\|_{L^q(\Omega_\varepsilon)} + \|f_{k,\varepsilon}\|_{L^q(\Omega_\varepsilon)} + \|g(w)\|_{L^q(\Omega_\varepsilon)}).$$

Here C is a constant depending only on Ω , p and k , while $\tilde{D}(n,q)$ depends only on n and q . The factor $1/\varepsilon^{\tilde{D}(n,q)}$, which does not appear in [1], is due to the scaling of Ω_ε . The uniform boundedness of $u_{k,\varepsilon}$, see equation (50), has also been used.

Assuming $w \in L^\infty(\Omega_\varepsilon)$, using the Sobolev embedding Theorem, and iterating (89) a finite number of times (starting with $q = 2$) we get

$$(90) \quad \|\tilde{w}\|_{L^\infty(\Omega_\varepsilon)} \leq \frac{C}{\varepsilon^{\tilde{D}(n)}} (\|\tilde{w}\|_{L^2(\Omega_\varepsilon)} + \|f_{k,\varepsilon}\|_{L^\infty(\Omega_\varepsilon)} + \|g(w)\|_{L^\infty(\Omega_\varepsilon)}),$$

for some other constant $\tilde{D}(n)$. Using inequalities (81), (82) and the results in Section 3 we have

$$(91) \quad \|\tilde{w}\|_{L^\infty(\Omega_\varepsilon)} \leq \begin{cases} C\varepsilon^{-\tilde{D}(n)} \left(\|\tilde{w}\|_{L^2(\Omega_\varepsilon)} + \varepsilon^{k+1} + \|w\|_{L^\infty(\Omega_\varepsilon)}^p \right) & \text{for } p \leq 2, \\ C\varepsilon^{-\tilde{D}(n)} \left(\|\tilde{w}\|_{L^2(\Omega_\varepsilon)} + \varepsilon^{k+1} + \|w\|_{L^\infty(\Omega_\varepsilon)}^2 \right) & \text{for } p > 2; \end{cases} \quad \|w\| \leq 1;$$

where C is a constant depending on Ω , p and k .

Now, in addition to (87), we require

$$(92) \quad d > \tilde{D}(n) \left(1 + \frac{1}{\min\{1, p\}} \right),$$

and we choose a real number s satisfying

$$(93) \quad s \in (0, d - \tilde{D}(n)); \quad s \min\{1, p - 1\} > \tilde{D}(n).$$

We also set

$$\tilde{\mathcal{B}}_j = \{w \in H^1(\Omega_\varepsilon) : \|w\|_{H^1(\Omega_\varepsilon)} \leq \varepsilon_j^d, \|w\|_{L^\infty(\Omega_\varepsilon)} \leq \varepsilon_j^s\},$$

and we endow $\mathcal{B}_j$ with the norm of $H^1(\Omega_\varepsilon)$.

For $K > 0$, we define a smooth positive function $Y_K : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$Y_K(t) = |t|^{p+1} \quad \text{for } |t| \leq K; \quad Y_K(t) = (K+1)^{p+1} \quad \text{for } |t| \geq K+1.$$

Let $I_{\varepsilon,K} : H^1(\Omega_\varepsilon) \rightarrow \mathbb{R}$ be the functional obtained substituting $|u|^{p+1}$ with $Y_K(u)$ in I_ε , and let $K_0 = 2 \sup u_{k,\varepsilon}$. Note that the functional $I_{\varepsilon,K}$ is well-defined on $H^1(\Omega_\varepsilon)$ since it contains a subcritical nonlinearity. Moreover, by the boundedness of $u_{k,\varepsilon}$, $I'_{\varepsilon,K}(u_{k,\varepsilon})$ and $I''_{\varepsilon,K}(u_{k,\varepsilon})$ coincide with $I'_\varepsilon(u_{k,\varepsilon})$ and $I''_\varepsilon(u_{k,\varepsilon})$ respectively. We also denote by F_{ε,K_0} the counterpart of the operator F_ε .

Then by (85), (86), (91) and the choice of s , the map F_{ε,K_0} is a contraction in $\tilde{\mathcal{B}}_j$. In fact, the norm estimates have already been verified before and give (omitting the index j)

$$\|\tilde{w}\|_{L^2(\Omega_\varepsilon)} \leq C\varepsilon^d.$$

Finally, using (87), (91), (92), (93) and the last inequality, we get

$$\begin{aligned} \|\tilde{w}\|_{L^\infty(\Omega_\varepsilon)} &\leq C\varepsilon^{-\tilde{D}(n)} \left(\varepsilon^d + \varepsilon^{k+1} + \varepsilon^{s(p \wedge 2)} \right) \\ &\leq C \left(\varepsilon^{d-\tilde{D}(n)} + \varepsilon^{k+1-\tilde{D}(n)} + \varepsilon^s \varepsilon^{s((p-1) \wedge 1) - \tilde{D}(n)} \right) \leq C\varepsilon^s. \end{aligned}$$

Hence there is a unique critical point of I_{ε,K_0} of the form $u_\varepsilon \in u_{k,\varepsilon} + \tilde{\mathcal{B}}_j$. Since there holds

$$\|u\|_\infty < K_0 \quad \text{for all } u \in u_{k,\varepsilon} + \tilde{\mathcal{B}}_j,$$

u_ε is a bounded solution of

$$(P_\varepsilon) \quad \begin{cases} -\varepsilon^2 \Delta u + u = |u|^{p-1}u & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

The positivity of u_ε can be achieved reasoning as in Subsection 5.1. This concludes the proof.

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